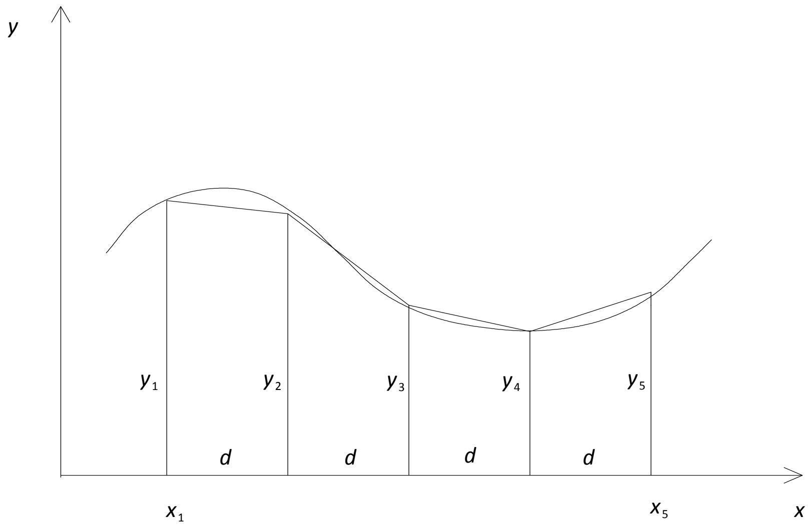
The trapezium rule

Sometimes it is not possible to find the area under the curve $y = f(x)$ by using integration techniques because you cannot integrate the function $f(x)$. In such cases we can find an approximation to the integral by using a numerical method. The method that we will look at here is called the trapezium rule. The technique involves approximating the area under the curve by finding the sum of the areas of several trapezia.

In the following diagram, we have split the area under the curve into trapeziums.



The area of a trapezium is given by the sum of the lengths of the two parallel sides, multiplied by the distance between them, divided by 2, *ie* it is the base multiplied by the average height.

Therefore the total area of the trapezia shown in the diagram (*ie* the estimate of the area under the curve between x_1 and x_5) is:

$$\frac{1}{2}(y_1 + y_2)d + \frac{1}{2}(y_2 + y_3)d + \frac{1}{2}(y_3 + y_4)d + \frac{1}{2}(y_4 + y_5)d$$

This can be rewritten as:

$$\frac{1}{2}(y_1 + 2y_2 + 2y_3 + 2y_4 + y_5)d$$

This can also be expressed in words:

> 'The sum of the first and last ordinate (or y value), plus twice the sum of the other ordinates, multiplied by $\frac{1}{2}d$.'

We can choose the number of ordinates to suit the area we are trying to estimate, and obviously the smaller the value of d, the more accurate the approximation will be.

Example

Use the trapezium rule to estimate the area under the curve $y = x^2$ between $x = 1$ and $x = 3$. Use 9 ordinates. Also find the true value of the area and comment on your answers.

Solution

Using 9 ordinates means that we need to split the area into 8 sections, each of width 0.25. The trapezium rule will then give the approximate area to be:

$$\frac{1}{2}\left(1^2 + 3^2 + 2(1.25^2 + 1.5^2 + 1.75^2 + 2^2 + 2.25^2 + 2.5^2 + 2.75^2)\right) \times 0.25$$

$$= 8.6875$$

The true area is:

$$\int_1^3 x^2 \, dx = \left[\frac{1}{3}x^3\right]_1^3 = \frac{1}{3}(3^3 - 1^3) = 8.67$$

You can see that here the trapezium rule has slightly over estimated the true area under the curve. If you plotted an accurate graph of the function $y = x^2$ and drew on the trapeziums you should be able to see that the over-estimation has occurred due to the shape of the curve.

Question 1.1

Estimate the area under the curve $y = \frac{2}{x}$ between $x = 2$ and $x = 3$, using 5 ordinates.

Will this be an overestimate or an underestimate of the true value?

Solutions

Solution 1.1

Using 5 ordinates means that we need to split the area into 4 sections, each of width 0.25. The trapezium rule will then give the approximate area to be:

$$\frac{1}{2}\left(\frac{2}{2}+\frac{4}{2.25}+\frac{4}{2.5}+\frac{4}{2.75}+\frac{2}{3}\right)\times 0.25=0.812$$

This will give an over estimate of the true value since the gradient of the curve is decreasing over the range of values.

We can check the exact value by integrating:

$$\int_2^3 \frac{2}{x}\,dx=\left[2\ln|x|\right]_2^3=2\ln 3-2\ln 2=0.811$$