Writing this in terms of the original variable x:

$$\int 3xe^{x^2}\,dx = \frac{3}{2}e^{x^2} + c$$ ♦♦

Integration by parts

Integration by parts gives us a method for integrating products of functions.

We can obtain the formula for integration by parts by starting with the product rule for differentiation, which was introduced in Chapter 6. This tells us that if u and v are functions of x, then:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

We can rearrange this to give:

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

The first term on the right-hand side is the derivative of the function uv with respect to x. If we integrate this with respect to x, we obtain uv, as integration is the reverse of differentiation. So, integrating the equation above with respect to x gives:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

This is the formula for integration by parts. It can be presented in the form of indefinite integration as above, or in the form of definite integration, with lower limit a and upper limit b, as shown in the box below. The basic procedure is the same whether the integrals are definite or indefinite, but for indefinite integration, the final answer will need to include a constant of integration.

Integration by parts

If u and v are functions of x, then:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

and

$$\int_a^b u\frac{dv}{dx}dx = \left[uv\right]_a^b - \int_a^b v\frac{du}{dx}dx$$

Looking at the formula for integration by parts in more detail, we see that on the left-hand side, the product to be integrated is split down into two parts, u and $\frac{dv}{dx}$. The right-hand side features u, v and $\frac{du}{dx}$. Of these, we already know what u is, as it is one of the functions on the left-hand side, but we will have to work out v and $\frac{du}{dx}$.

This helps us when choosing which function to associate with u in the formula, and which to associate with $\frac{dv}{dx}$: we should try to make these associations so that we can easily *differentiate* the function u and easily *integrate* the function $\frac{dv}{dx}$.

Another issue that affects our choice of u and $\frac{dv}{dx}$ is the fact that the right-hand side of the formula for integration by parts features:

$$\int v \frac{du}{dx} dx$$

The process of integration by parts is therefore only useful if the integral we are faced with on the right-hand side is easier to work out than the one we started with on the left-hand side, and ensuring this is the case also affects our choice of u and $\frac{dv}{dx}$.

It is not always straightforward to decide which function to associate with u (the function to differentiate) and which to associate with $\frac{dv}{dx}$ (the function to integrate), but it does get easier with practice. We'll take a look at some examples to highlight the issues.

Example 7.14

Integrate xe^x with respect to x.

Solution

Both x and e^x can be easily differentiated and integrated, so we should choose u and $\frac{dv}{dx}$ so that the resulting integral is easier than the original one. One thing to note here is that if we set u to be a linear function of x, then its derivative will be a constant, which is likely to make the integral on the right-hand side easier to deal with.

Letting $u = x$ and $\frac{dv}{dx} = e^x$, so that $\frac{du}{dx} = 1$ and $v = e^x$, we have:

$$\int \underbrace{x}_{u} \underbrace{e^x}_{\frac{dv}{dx}} dx = \underbrace{x}_{u} \underbrace{e^x}_{v} - \int \underbrace{1}_{\frac{du}{dx}} \underbrace{e^x}_{v} dx$$

Carrying out the integration on the right-hand side gives:

$$\int xe^x dx = xe^x - e^x + c = (x-1)e^x + c$$ ♦♦

Let's see what would have happened in Example 7.14 if we had chosen u and $\frac{dv}{dx}$ to be the other way around, so $u = e^x$ and $\frac{dv}{dx} = x$. In this case, $\frac{du}{dx} = e^x$ and $v = \frac{1}{2}x^2$, so the formula for integration by parts gives:

$$\int \underbrace{x}_{\frac{dv}{dx}} \underbrace{e^x}_{u} dx = \underbrace{e^x}_{u} \times \underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{e^x}_{\frac{du}{dx}} \times \underbrace{\frac{1}{2}x^2}_{v} dx$$