Induction

One method of proving a general mathematical result is the method of (mathematical) induction. To prove that a result is true for all positive integers, we prove that if the result is true for any particular integer *k* then it must also be true for the next integer $k+1$. If we can also show that it is true when $k=1$, then it must be true for all positive integers $k=1,2,\ldots$.

Example

Prove by induction that $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$.

Solution

Assume the result is true for $n=k$, *ie*:

$$1+2+3+\cdots+k=\tfrac{1}{2}k(k+1)$$

Adding the next term on to both sides it follows that:

$$\begin{aligned}1+2+3+\cdots+k+(k+1)&=\tfrac{1}{2}k(k+1)+(k+1)\\&=\tfrac{1}{2}(k+1)(k+2)\\&=\tfrac{1}{2}(k+1)\left[(k+1)+1\right]\end{aligned}$$

Since this is what the original equation 'predicts' when $n=k+1$, we have shown that if the result is true for $n=k$ then it is also true for $n=k+1$.

Consider whether the result is true when $n=1$:

LHS = 1 RHS = 1

So the result is true for $n=1$ and by the above result it must also be true for $n=2,3,4,\ldots$ *ie* for all positive integer values of *n*.

Question 1.1

Prove by induction that $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

Solutions

Solution 1.1

Assume the result is true for $n=k$, *ie*:

$$1^2+2^2+3^2+\cdots+k^2=\tfrac{1}{6}k(k+1)(2k+1)$$

Adding the next term on to both sides:

$$\begin{aligned}1^2+2^2+3^2+\cdots+k^2+(k+1)^2&=\tfrac{1}{6}k(k+1)(2k+1)+(k+1)^2\\&=\tfrac{1}{6}(k+1)[k(2k+1)+6(k+1)]\\&=\tfrac{1}{6}(k+1)(2k^2+k+6k+6)\\&=\tfrac{1}{6}(k+1)(2k^2+7k+6)\\&=\tfrac{1}{6}(k+1)(k+2)\left(2k+3\right)\\&=\tfrac{1}{6}(k+1)[(k+1)+1][2(k+1)+1]\end{aligned}$$

So we have shown that if the result is true for $n=k$ then it is also true for $n=k+1$.

Consider when $n=1$:

LHS = 1 RHS = 1

So the result is true for $n=1$ and by mathematical induction it is also true for *n* = 2, 3, 4, … *ie* for all positive integer values of *n*.