Differentiating an integral (Leibniz's formula)

Here we use the result that $\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt$, where *a* and *b* are constants.

This can be thought of as taking the $\frac{d}{dx}$ inside the integral. This can be generalised to:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = b'(x)f(x,b(x)) - a'(x)f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt$$

This formula can be found on page 3 of the *Tables*.

The formula is very useful in cases where the integral cannot easily be evaluated directly, for example $\frac{d}{dx}\int_0^\infty e^{-xt^2}\,dt$.

 © IFE: 2019 Examinations

Example

Evaluate $\frac{d}{dx}\int_0^x x^2 + t\,dt$.

Solution

Here $a(x)=0$, $b(x)=x$, $f(x,t)=x^2+t$, so:

$$\frac{d}{dx}\int_0^x x^2 + t\ dt = 1(x^2+x)-0+\int_0^x 2x\ dt$$

$$= x^2 + x + 2x^2$$

$$= 3x^2 + x$$

In this case we can show that this is the same as integrating directly:

$$\int_0^x x^2 + t\,dt = \left[x^2 t + \frac{1}{2}t^2\right]_0^x = x^3 + \frac{1}{2}x^2$$

So $\frac{d}{dx}\int_0^x x^2 + t\,dt = \frac{d}{dx}\left(x^3 + \frac{1}{2}x^2\right) = 3x^2 + x$.

Question 1.1

Evaluate $\frac{d}{dx}\int_0^{2x+3}\left[(x+1)^2 + tx\right]\,dt$.

Solutions

Solution 1.1

We need to find $\dfrac{d}{dx}\int_0^{2x+3} (x+1)^2 + tx \; dt$, so comparing with the general equation:

$$a(x)=0,\ b(x)=2x+3,\ \text{and}\ f(x,t)=(x+1)^2+tx$$

giving:

$$a'(x)=0,\ b'(x)=2,\ \text{and}\ \frac{\partial f}{\partial x}=2(x+1)+t$$

So:

$$\begin{aligned}\frac{d}{dx}\int_0^{2x+3}(x+1)^2+tx\;dt &= 2\Big[(x+1)^2+x(2x+3)\Big]-0+\int_0^{2x+3}2(x+1)+t\;dt\\ &= 2(3x^2+5x+1)+\Big[2(x+1)t+\tfrac{1}{2}t^2\Big]_0^{2x+3}\\ &= 2(3x^2+5x+1)+\Big(2(x+1)(2x+3)+\tfrac{1}{2}(2x+3)^2\Big)\\ &= 12x^2+26x+12\tfrac{1}{2}\end{aligned}$$

 © IFE: 2019 Examinations