Differential equations

Any equation which contains $\frac{d^n y}{dx^n}$ in some form, for example, $\frac{dy}{dx} = 2x + 3y$, is called a differential equation. Here we are going to consider some of the many techniques used to *solve* these equations. Solving differential equations means finding $y = f(x)$.

Solution by direct integration

If we integrate $\frac{dy}{dx}$, then we get *y* and if we integrate $\frac{d^2 y}{dx^2}$, then we get $\frac{dy}{dx}$, *etc*. This enables us to solve differential equations of the form $\frac{d^n y}{dx^n} = f(x)$.

Example

Solve the differential equation $\frac{d^2 r}{dt^2} = 2t^2$.

Solution

Integrating once with respect to *t*:

$$\frac{dr}{dt} = \tfrac{2}{3}t^3 + c$$

Integrating again:

$$r = \tfrac{1}{6}t^4 + ct + d$$

As seen in this example, we can end up with several constants in the solution. This is called a 'general solution'. You can find a 'particular solution' if there are 'boundary conditions' given in the question, giving you particular values of the variables involved.

Example

Solve the differential equation $\frac{dy}{dx} = e^x$, given that $y = 1$ when $x = \ln 2$.

Solution

The general solution of the differential equation is $y = e^x + c$, but we know that when $x = \ln 2,\ y = 1$.

So:

$$1 = 2 + c \Rightarrow c = -1$$

and the particular solution is:

$$y = e^x - 1$$

Question 1.1

Find the general solution of the differential equation $\frac{d^2x}{dt^2} = at^2$. Given that when $t = 0,\ x = 4$, when $t = 1,\ x = \frac{97}{12}$, and when $t = -1,\ x = \frac{1}{12}$, find a and the particular solution.

Solution by separation of variables

If $\frac{dy}{dx} = f(x,y)$, and $f(x,y)$ can be expressed as $p(x)q(y)$, then the differential equation can be solved by separating the variables, *ie* by writing it in the form:

$$\int \frac{1}{q(y)}\,dy = \int p(x)\,dx$$

Example

Solve the differential equation $\frac{dr}{dt} = \frac{r}{2t+1}$, where $r > 0$ and $t > 0$.

Solution

We can rearrange this in the form $\frac{dr}{r} = \frac{dt}{2t+1}$, so:

$$\int \frac{1}{r}\,dr = \int \frac{1}{2t+1}\,dt$$

$$\ln r = \tfrac{1}{2}\ln(2t+1) + c \quad \Rightarrow \quad \ln r = \ln\left(\sqrt{2t+1}\right) + c$$

Note that we do not need to include modulus signs in the integrated expressions as both r and $2t+1$ are positive. Note also that we need to include a constant on one side of the equation, not both.

Taking exponentials of both sides:

$$e^{\ln r} = e^{\ln\left(\sqrt{2t+1}\right)+c} \quad \Rightarrow \quad r = e^c\sqrt{2t+1} \quad \Rightarrow \quad r = k\sqrt{2t+1}$$

where $k = e^c$.

Question 1.2

Find the particular solution of the differential equation $(x^2+9)\frac{dy}{dx} = xy + 2x$, where $y > 0$, given that when $x = 4, y = 0.5$.

Question 1.3

Find the general solution to the differential equation $\frac{dN}{dt} = (a - bN)N$, where $0 < N < \frac{a}{b}$.
This is known as the logistic equation or the Verhulst equation after the mathematician who used it in his study of populations.

Solution by integrating factor

A differential equation of the form $\frac{dy}{dx}+f(x)y=g(x)$ can be solved by use of an integrating factor. The integrating factor here is $\exp\left[\int f(x)\,dx\right]$, and the solution of the differential equation is then the solution of:

$$y\exp\left[\int f(x)\,dx\right]=\int g(x)\exp\left[\int f(x)\,dx\right]dx$$

Example

Find the general solution of $\frac{dy}{dx}+4y=2e^{3x}$.

Solution

The integrating factor is $\exp\left[\int 4\,dx\right]=e^{4x}$. So we multiply through by e^{4x} which gives $e^{4x}\frac{dy}{dx}+4ye^{4x}=2e^{7x}$.

We see that the LHS is the derivative of ye^{4x}, derived from the product rule. So $\frac{d}{dx}(ye^{4x})=2e^{7x}$.

We then need to solve $ye^{4x}=\int 2e^{7x}dx$, which gives $ye^{4x}=\frac{2}{7}e^{7x}+c$. The solution of our differential equation is $y=\frac{2}{7}e^{3x}+ce^{-4x}$, where c is an arbitrary constant.

Question 1.4

Find the particular solution of the differential equation $3xy\frac{dy}{dx}=y^2+2xy$, where $x>0$, given that $y=1$ when $x=27$.

Solutions

Solution 1.1

Integrating twice gives:

$$\frac{dx}{dt}=\frac{at^3}{3}+c$$

$$x=\frac{at^4}{12}+ct+d$$

But using the boundary conditions:

$$t=0 \Rightarrow d=4$$

$$t=1 \Rightarrow \frac{97}{12}=\frac{a}{12}+c+4$$

$$t=-1 \Rightarrow \frac{1}{12}=\frac{a}{12}-c+4$$

Solving these simultaneously, we get $a=1$, $c=4$, so the particular solution is:

$$x=\frac{t^4}{12}+4t+4$$

Solution 1.2

Separating the variables gives us:

$$\int \frac{1}{y+2}\,dy=\int \frac{x}{x^2+9}\,dx$$

Integrating both sides gives us:

$$\ln(y+2)=c+\tfrac{1}{2}\ln(x^2+9)$$

Note that no modulus signs are required here as x^2+9 is always positive, and $y+2$ is always positive since we are given that $y>0$.

So:

$$e^{\ln(y+2)} = e^{c+\ln\sqrt{x^2+9}} \;\Rightarrow\; y+2 = e^c\sqrt{x^2+9} \;\Rightarrow\; y = k\sqrt{x^2+9}-2$$

where $k = e^c$. But if $y = 0.5$ when $x = 4$, then $k = 0.5$, so the particular solution is:

$$y = 0.5\sqrt{x^2+9}-2$$

Solution 1.3

The differential equation can be solved by separating the variables:

$$\int \frac{dN}{(a-bN)N} = \int dt$$

$$\int \frac{1/a}{N} + \frac{b/a}{a-bN}\, dN = \int dt$$

$$\frac{1}{a}\ln N - \frac{1}{a}\ln(a-bN) + c = t$$

Note that $N > 0$ and $a-bN > 0$ from the condition in the question, so no modulus signs are required here.

Setting $c = \frac{1}{a}\ln k$, for some positive constant k, this can be simplified to give the general solution:

$$\frac{1}{a}\ln\frac{kN}{a-bN} = t \;\Rightarrow\; \frac{kN}{a-bN} = e^{at} \;\Rightarrow\; N = \frac{ae^{at}}{k+be^{at}}$$

Solution 1.4

The differential equation can be rewritten in the form $\frac{dy}{dx}-\frac{1}{3x}y=\frac{2}{3}$, providing that $y\neq 0$. $y=0$ is a trivial solution to the differential equation.

Using this version, we can see that the integrating factor is
$\exp\left[\int -\frac{1}{3x}dx\right]=\exp\left[-\frac{1}{3}\ln x\right]=x^{-\frac{1}{3}}$.

Note that since $x>0$, we don't need to include modulus signs here.

Multiplying through by the integrating factor and integrating both sides with respect to x gives the general solution to the differential equation:

$$yx^{-\frac{1}{3}}=\int \frac{2}{3}x^{-\frac{1}{3}}\,dx=x^{\frac{2}{3}}+c$$

But we know that $y=1$ when $x=27$, so:

$$\frac{1}{3}=9+c \quad\Rightarrow\quad c=-8\tfrac{2}{3}$$

So the particular solution is $yx^{-\frac{1}{3}}=x^{\frac{2}{3}}-8\tfrac{2}{3}$ or $y=x-8\tfrac{2}{3}x^{\frac{1}{3}}$.