



# Probability, Third Edition

By David J Carr & Michael A Gauger

Published by BPP Professional Education

## Solutions to practice questions – Chapter 6

### Solution 6.1

Let the losses due to storm, fire, and theft be denoted  $X_1$ ,  $X_2$  and  $X_3$  respectively.

$$\begin{aligned} \Pr(\max\{X_1, X_2, X_3\} > 2.5) &= 1 - \Pr(\max\{X_1, X_2, X_3\} \leq 2.5) \\ &= 1 - \Pr(X_1 \leq 2.5) \times \Pr(X_2 \leq 2.5) \times \Pr(X_3 \leq 2.5) \end{aligned}$$

The distribution function for a uniform distribution with range  $[a, b]$  is:

$$F(x) = \frac{x - a}{b - a}$$

So, we can calculate the probability as:

$$\begin{aligned} \Pr(\max\{X_1, X_2, X_3\} > 2.5) &= 1 - \Pr(X_1 \leq 2.5) \times \Pr(X_2 \leq 2.5) \times \Pr(X_3 \leq 2.5) \\ &= 1 - (1) \left(\frac{2.5}{3}\right) \left(\frac{2.5}{4}\right) = 1 - 0.521 = 0.479 \end{aligned}$$

### Solution 6.2

We have:

$$\begin{aligned} E[X] &= \frac{a + b}{2} = 1,879 \\ \text{var}(X) &= \frac{(b - a)^2}{12} = 507 \Rightarrow b - a = \sqrt{12 \times 507} = 78 \end{aligned}$$

Hence:

$$\begin{aligned} a + b &= 2 \times 1,879 = 3,758 \\ \Rightarrow a + (78 + a) &= 3,758 \\ \Rightarrow a &= 1,840 \\ \Rightarrow b &= 1,918 \end{aligned}$$

**Solution 6.3**

Since we have formulas for  $E[X]$  and  $E[X^2]$ , it is convenient to first calculate the second moment:

$$E[X^2] = \text{var}(X) + (E[X])^2 = 18.75 + 2.5^2 = 25$$

Hence:

$$2.5 = E[X] = \frac{\theta}{\alpha - 1}$$

$$25 = E[X^2] = \frac{\theta^2 2!}{(\alpha - 1)(\alpha - 2)}$$

We solve these simultaneous equations to find the parameters:

$$\frac{25}{2.5^2} = \frac{E[X^2]}{(E[X])^2} = \frac{\theta^2 2!}{(\alpha - 1)(\alpha - 2)} \times \left(\frac{\alpha - 1}{\theta}\right)^2 = \frac{2(\alpha - 1)}{\alpha - 2}$$

$$\Rightarrow \alpha = 3, \theta = 5$$

Now it is easy to calculate the 90th percentile:

$$0.9 = F(x_{0.9}) = 1 - \left(\frac{\theta}{x_{0.9} + \theta}\right)^\alpha = 1 - \left(\frac{5}{x_{0.9} + 5}\right)^3$$

$$\Rightarrow x_{0.9} = 5 \left(0.1^{-\frac{1}{3}} - 1\right) = 5.772$$

**Solution 6.4**

If we notice that  $X$  follows a Pareto distribution with parameters  $\alpha = 3$  and  $\theta = 2,000$ , the expected return is easily calculated using the general result as:

$$E[X] = \frac{\theta}{\alpha - 1} = \frac{2,000}{3 - 1} = 1,000$$

**Solution 6.5**

We want to calculate the following:

$$\Pr(X < 2 | X > 1.5) = \frac{\Pr(X < 2 \cap X > 1.5)}{\Pr(X > 1.5)} = \frac{\Pr(1.5 < X < 2)}{\Pr(X > 1.5)} = \frac{F(2) - F(1.5)}{1 - F(1.5)}$$

Note that the pdf is that of a one-parameter Pareto distribution with  $\alpha = 3$  and  $\theta = 1$ , so the cdf is:

$$F(x) = 1 - x^{-3} \quad \text{for } x > 1$$

Finally:

$$\Pr(X < 2 | X > 1.5) = \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{(1 - 2^{-3}) - (1 - 1.5^{-3})}{1.5^{-3}} = 0.5781$$

**Solution 6.6**

The lifetime  $X$  follows a single-parameter Pareto distribution with  $\alpha = 3$  and  $\theta = 3$ , as detailed on page 138.

Recall that we can define  $X = Y + \theta$ , where  $Y$  follows a standard (two-parameter) Pareto distribution with  $\alpha = 3$  and  $\theta = 3$ .

The expected value of  $X$  is given by:

$$E[X] = E[Y + \theta] = E[Y] + \theta = \frac{\theta}{\alpha - 1} + \theta = \frac{\alpha\theta}{\alpha - 1}$$

Hence, the expected value is:

$$E[X] = \frac{\alpha\theta}{\alpha - 1} = \frac{3 \times 3}{2} = 4.5$$

**Solution 6.7**

For the exponential distribution with mean 50, we have:

$$\Pr(X > x) = 1 - F(x) = e^{-x/50}$$

Hence:

$$\Pr(X > 200 | X > 50) = \frac{\Pr(X > 200)}{\Pr(X > 50)} = \frac{e^{-200/50}}{e^{-50/50}} = \frac{e^{-4}}{e^{-1}} = e^{-3} = 0.0498$$

Alternatively, using the memoryless property we have:

$$\Pr(X > 200 | X > 50) = \Pr(X > 150) = e^{-150/50} = e^{-3} = 0.0498$$

**Solution 6.8**

Let  $X$  be the random time (in days) until the high-risk driver has an accident. Since  $X$  follows an exponential distribution, the cdf is:

$$F(x) = 1 - e^{-x/\theta}$$

We are given that

$$\begin{aligned} 0.30 &= \Pr(X < 50) = 1 - e^{-50/\theta} \\ \Rightarrow e^{-50/\theta} &= 0.70 \end{aligned}$$

Hence:

$$\Pr(X < 80) = 1 - e^{-80/\theta} = 1 - \left(e^{-50/\theta}\right)^{8/5} = 1 - 0.70^{8/5} = 0.4349$$

**Note:** It is unnecessary to calculate the precise value of  $\theta$ . If you did solve for  $\theta$ , you should find that  $\theta = 140.184$ .

**Solution 6.9**

Let  $T$  denote the random time required to repair the machine. It is assumed to be exponentially distributed with mean  $\theta_T = 2$ . Let  $X$  denote the cost of replacement parts. It is assumed to be gamma distributed with parameters  $\alpha$  and  $\theta_X$  such that:

$$100 = E[X] = \alpha \theta_X$$

$$5,000 = \text{var}(X) = \alpha \theta_X^2$$

Solving these equations results in:

$$\alpha = 2 \quad \theta_X = 50$$

We are asked to calculate the probability of the event  $\{T > 3\} \cup \{X > 150\}$ .

We can calculate the two components as follows:

$$\Pr(T > 3) = 1 - \Pr(T < 3) = e^{-3/2} = 0.2231$$

$$\Pr(X > 150) = 1 - \Pr(X < 150) = e^{-150/50} \left( 1 + \frac{150}{50} \right) = 0.1991$$

By the additive probability law, we have:

$$\begin{aligned} \Pr(\{T > 3\} \cup \{X > 150\}) &= \Pr(T > 3) + \Pr(X > 150) - \Pr(\{T > 3\} \cap \{X > 150\}) \\ &= \Pr(T > 3) + \Pr(X > 150) - \Pr(T > 3)\Pr(X > 150) \\ &= 0.2231 + 0.1991 - 0.2231 \times 0.1991 = 0.3778 \end{aligned}$$

**Solution 6.10**

We can calculate the required probability as follows:

$$\Pr(4 < S < 8) = \Pr(4 < S < 8 \cap N = 0) + \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1)$$

If there are no claims (*ie*  $N = 0$ ), then the claim amount must be zero, so:

$$\Pr(4 < S < 8 \cap N = 0) = 0$$

Therefore:

$$\Pr(4 < S < 8) = \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1)$$

So from basic laws of probability, we have:

$$\begin{aligned} \Pr(4 < S < 8) &= \Pr(4 < S < 8 \cap N = 1) + \Pr(4 < S < 8 \cap N > 1) \\ &= \Pr(4 < S < 8 | N = 1) \times \Pr(N = 1) + \Pr(4 < S < 8 | N > 1) \times \Pr(N > 1) \\ &= \underbrace{(F(8) - F(4))}_{\text{exp, } \theta=5} \times \frac{1}{3} + \underbrace{(F(8) - F(4))}_{\text{exp, } \theta=8} \times \frac{1}{6} \\ &= \left( (1 - e^{-8/5}) - (1 - e^{-4/5}) \right) \times \frac{1}{3} + \left( (1 - e^{-8/8}) - (1 - e^{-4/8}) \right) \times \frac{1}{6} \\ &= 0.1223 \end{aligned}$$

**Solution 6.11**

The waiting time between accidents follows an exponential distribution with mean  $\theta = \frac{5}{2} = 2.5$  days.

Hence:

$$\Pr(X > 3) = 1 - F(3) = e^{-\frac{3}{2.5}} = e^{-1.2} = 0.3012$$

**Solution 6.12**

From the form of the pdf, we can see that  $X$  follows a gamma distribution with  $\alpha = 6$  and  $\theta = 100$ .

Hence:

$$E[X] = \alpha\theta = 6 \times 100 = 600$$

**Solution 6.13**

First, we'll determine the parameter values:

$$15 = E[X] = \alpha\theta$$

$$75 = \text{var}(X) = \alpha\theta^2$$

$$\Rightarrow \theta = \frac{\text{var}(X)}{E[X]} = 5 \Rightarrow \alpha = 3$$

Since  $\alpha$  is a positive integer, the cdf can be written as:

$$F(x) = 1 - e^{-x/5} \left( 1 + \frac{x}{5} + \frac{x^2}{2!5^2} \right)$$

Finally:

$$\Pr(X > 30 | X > 15) = \frac{\Pr(X > 30)}{\Pr(X > 15)} = \frac{1 - F(30)}{1 - F(15)} = \frac{1 - 0.93803}{1 - 0.57681} = \frac{0.06197}{0.42319} = 0.1464$$

**Solution 6.14**

From the form of the moment generating function, we can see that  $X$  follows a gamma distribution with parameters  $\alpha = 2$  and  $\theta = 3$ . Hence:

$$\text{var}(X) = \alpha\theta^2 = 2 \times 3^2 = 18$$

**Solution 6.15**

Differentiating the cumulant generating function, we have:

$$R_X(t) = \ln M_X(t) = \ln\left((1-\theta t)^{-\alpha}\right) = -\alpha \ln(1-\theta t)$$

$$R'_X(t) = \frac{\alpha\theta}{(1-\theta t)}$$

$$R''_X(t) = \frac{\alpha\theta^2}{(1-\theta t)^2} \Rightarrow R''_X(0) = \alpha\theta^2$$

$$R'''_X(t) = \frac{2\alpha\theta^3}{(1-\theta t)^3} \Rightarrow R'''_X(0) = 2\alpha\theta^3$$

Hence the skewness is:

$$\frac{E[(X-\mu)^3]}{\sigma^3} = \frac{R'''_X(0)}{(R''_X(0))^{3/2}} = \frac{2\alpha\theta^3}{\alpha^{3/2}\theta^3} = \frac{2}{\sqrt{\alpha}}$$

**Solution 6.16**

Let  $X$  be the sum of the two sample values.

By the additive property, the sum of the two sample values follows a chi-square distribution with 2 degrees of freedom. A chi-square distribution with 2 degrees of freedom is equivalent to an exponential distribution with parameter  $\theta = 2$ .

Hence the required probability is:

$$\Pr(\text{Sample mean} < 1.15) = \Pr(X < 2.3) = F(2.3) = 1 - e^{-2.3/2} = 0.6834$$

**Solution 6.17**

With  $\tau = 0.25$  and  $\theta = 10$ , we have:

$$E[X] = \theta\Gamma\left(1 + \frac{1}{\tau}\right) = 10\Gamma(5) = 10 \times 4! = 240$$

$$\text{var}(X) = \frac{\theta^2}{\tau} \left[ 2\Gamma\left(\frac{2}{\tau}\right) - \frac{1}{\tau} \left( \Gamma\left(\frac{1}{\tau}\right) \right)^2 \right] = \frac{10^2}{0.25} \left[ 2\Gamma(8) - 4(\Gamma(4))^2 \right] = 3,974,400$$

**Solution 6.18**

The given pdf is for a Weibull distribution with  $\tau = 3$  and  $\theta = \sqrt[3]{\frac{1}{4}}$ .

Using the general result for the cdf, we have:

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\tau} = 1 - e^{-4x^3}$$

$$\Rightarrow \Pr(X > 0.5) = 1 - F(0.5) = e^{-4 \times 0.5^3} = e^{-0.5} = 0.6065$$

**Solution 6.19**

With  $a = 1$  and  $b = 7$ , the pdf is:

$$f(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1} (1-x)^{b-1} = \frac{7!}{6!} (1-x)^6 = 7(1-x)^6$$

Hence, the cdf is:

$$F(x) = \int_0^x f(y) dy = \int_0^x 7(1-y)^6 dy = -(1-y)^7 \Big|_0^x = 1 - (1-x)^7$$

Now we can compute the median as:

$$0.5 = F(x_{0.5}) = 1 - (1 - x_{0.5})^7 \Rightarrow x_{0.5} = 1 - 0.5^{\frac{1}{7}} = 0.0943$$

**Solution 6.20**

The second moment is:

$$E[X^2] = \text{var}(X) + (E[X])^2 = \frac{1}{48} + \left(\frac{1}{4}\right)^2 = \frac{1}{12}$$

Solving the moment equations for the parameters, we have:

$$E[X] = \frac{a}{a+b} = \frac{1}{4} \Rightarrow b = 3a$$

$$E[X^2] = \left(\frac{a}{a+b}\right) \left(\frac{a+1}{a+b+1}\right) = \frac{1}{12} \Rightarrow \frac{a+1}{a+b+1} = \frac{1}{3}$$

Substituting, we have:

$$\frac{a+1}{4a+1} = \frac{1}{3} = \frac{3}{9} \Rightarrow a = 2 \Rightarrow b = 6$$