



# Probability, Third Edition

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## Solutions to practice questions – Chapter 5

### Solution 5.1

Let  $X$  be the number of visits (out of the next 10 visits) that result in a referral. Then  $X$  follows a binomial distribution with  $n = 10$  and  $p = 0.28$ , and the required probability is:

$$\Pr(X = 4) = {}_{10}C_4 (0.28^4) (0.72^6) = 0.1798$$

### Solution 5.2

Let  $X$  be the number of people who suffer a side effect. Then  $X$  follows a binomial distribution with  $n = 1,000$  and  $p = 0.005$ , and the required probability is:

$$\begin{aligned} \Pr(X \leq 1) &= \Pr(X = 0) + \Pr(X = 1) \\ &= 0.995^{1,000} + {}_{1,000}C_1 (0.005^1) (0.995^{999}) \\ &= 0.0067 + 0.0334 = 0.0401 \end{aligned}$$

### Solution 5.3

The probability that a report is filed is:

$$\Pr(\text{More than one injury}) = 0.23 + 0.17 + 0.09 + 0.04 = 0.53$$

Let  $X$  be the number of reports filed for the next 20 games. Then  $X$  follows a binomial distribution with  $n = 20$  and  $p = 0.53$ .

The expected number of reports is:

$$E[X] = np = 20 \times 0.53 = 10.6$$

The standard deviation of the number of reports is:

$$\sqrt{npq} = \sqrt{20 \times 0.53 \times 0.47} = 2.232$$

**Solution 5.4**

Using the moment generating function:

$$M_X(t) = (pe^t + q)^n$$

$$\Rightarrow M'_X(t) = (n)(pe^t)(pe^t + q)^{n-1}$$

$$\Rightarrow M''_X(t) = (n)(pe^t)(pe^t + q)^{n-1} + (n)(n-1)(pe^t)^2(pe^t + q)^{n-2} \quad (\text{product rule})$$

(i) The mean is:

$$E[X] = M'_X(0) = (n)(p)(p+q)^{n-1} = np \quad \text{since } p+q=1$$

(ii) The variance is:

$$\text{var}(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = M''_X(0) = (n)(p)(p+q)^{n-1} + (n)(n-1)(p)^2(p+q)^{n-2} = np + n(n-1)p^2$$

$$\Rightarrow \text{var}(X) = np + n(n-1)p^2 - (np)^2 = np - np^2 = np(1-p) = npq$$

**Solution 5.5**

The number of claims,  $X$ , follows the binomial distribution with  $n = 10$  and  $p = 0.125$ .

The expected number of claims is:

$$E[X] = np = 10 \times 0.125 = 1.25$$

The variance is:

$$\text{var}(X) = npq = 10 \times 0.125 \times 0.875 = 1.09375$$

The standard deviation is:

$$\sqrt{\text{var}(X)} = \sqrt{1.09375} = 1.04583$$

The reserve is:

$$10 \times (1.25 + 2 \times 1.04583) = 33.417 \text{ (\$million)}$$

**Solution 5.6**

The number of hurricanes,  $X$ , can be modeled using a binomial distribution if we treat each year as a Bernoulli trial with  $\text{Pr}(\text{Success}) = \text{Pr}(\text{Hurricane}) = p = 0.05$  and the number of trials (years) is  $n = 20$ .

Using the binomial distribution probability function:

$$\begin{aligned} \text{Pr}(X < 3) &= \text{Pr}(X = 0) + \text{Pr}(X = 1) + \text{Pr}(X = 2) \\ &= ({}_{20}C_0)(0.05^0)(0.95^{20}) + ({}_{20}C_1)(0.05^1)(0.95^{19}) + ({}_{20}C_2)(0.05^2)(0.95^{18}) \\ &= 0.35849 + 0.37735 + 0.18868 = 0.9245 \end{aligned}$$

**Solution 5.7**

Let's define  $X$  as the number of tourists who do show up. Then  $X$  has a binomial distribution with  $n = 21$  trials and with  $p = 0.98$  (we define "success" as a tourist showing up).

The tour operator's revenue,  $R$ , is:

- $21 \times 50 = 1,050$  if  $X \leq 20$  (a seat is available for everyone)
- $21 \times 50 - 100 = 950$  if all 21 tourists show up

The expected revenue is thus:

$$\begin{aligned} E[R] &= 1,050 \times \Pr(X \leq 20) + 950 \times \Pr(X = 21) \\ &= 1,050 \times (1 - \Pr(X = 21)) + 950 \times \Pr(X = 21) \\ &= 1,050 \times (1 - 0.98^{21}) + 950 \times 0.98^{21} = 985 \end{aligned}$$

**Solution 5.8**

Let  $X$  be the annual number of claims.

The probability of no more than 2 claims being filed in the next year is:

$$\begin{aligned} \Pr(X \leq 2) &= \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \\ &= p^r + \frac{r!}{1!(r-1)!} p^r q + \frac{(r+1)!}{2!(r-1)!} p^r q^2 \\ &= 0.7^8 + \frac{8!}{1!7!} \times 0.7^8 \times 0.3 + \frac{9!}{2!7!} \times 0.7^8 \times 0.3^2 \\ &= 0.0576 + 0.1384 + 0.1868 = 0.3828 \end{aligned}$$

**Solution 5.9**

Let  $X$  be the annual number of claims.

We must solve two simultaneous equations involving the two parameters:

$$3 = E[X] = \frac{rq}{p} \quad 7.5 = \text{var}(X) = \frac{rq}{p^2}$$

If we divide the first equation by the second, we see that:

$$p = \frac{E[X]}{\text{var}(X)} = \frac{3}{7.5} = 0.4 \Rightarrow q = 1 - p = 0.6 \Rightarrow r = 2$$

Hence the required probability is:

$$\Pr(X = 3) = \frac{(r+2)!}{3!(r-1)!} p^r q^3 = \frac{4!}{3!1!} \times 0.4^2 \times 0.6^3 = 0.13824$$

**Solution 5.10**

Using the moment generating function:

$$M_X(t) = \left( \frac{1 - qe^t}{p} \right)^{-r} = p^r (1 - qe^t)^{-r}$$

$$\Rightarrow M'_X(t) = (p^r)(-qe^t)(-r)(1 - qe^t)^{-r-1} = (rqp^r)(e^t)(1 - qe^t)^{-r-1}$$

$$\Rightarrow M''_X(t) = (rqp^r) \left[ (e^t)(1 - qe^t)^{-r-1} + (e^t)(-qe^t)(-r-1)(1 - qe^t)^{-r-2} \right] \quad (\text{product rule})$$

(i) The mean is:

$$E[X] = M'_X(0) = (rqp^r)(1)(1 - q)^{-r-1} = (rqp^r)(p)^{-(r+1)} = \frac{rq}{p}$$

(ii) The variance is:

$$\text{var}(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = M''_X(0) = (rqp^r) \left[ (1 - q)^{-(r+1)} + (q)(r+1)(1 - q)^{-(r+2)} \right] = \frac{rq}{p} + \frac{r(r+1)q^2}{p^2}$$

$$\Rightarrow \text{var}(X) = \frac{rq}{p} + \frac{r(r+1)q^2}{p^2} - \left( \frac{rq}{p} \right)^2 = \frac{rq}{p} + \frac{rq^2}{p^2} = \frac{rq(p+q)}{p^2} = \frac{rq}{p^2}$$

**Solution 5.11**

Let  $X$  be the number of accident-free months before the third month in which at least one accident occurs.

Then  $X$  follows a negative binomial distribution with  $p = 0.3$  and  $r = 3$ , and the required probability is:

$$\begin{aligned} \Pr(X \geq 3) &= 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2) \\ &= 1 - p^r - \frac{r!}{1!(r-1)!} p^r q - \frac{(r+1)!}{2!(r-1)!} p^r q^2 \\ &= 1 - 0.3^3 - \frac{3!}{1!2!} \times 0.3^3 \times 0.7 - \frac{4!}{2!2!} \times 0.3^3 \times 0.7^2 \\ &= 1 - 0.027 - 0.0567 - 0.07938 = 0.83692 \end{aligned}$$

**Solution 5.12**

The probability of success (ie all 5 coins are heads) on a single trial is:

$$\left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

Then  $N$  (the number of trials until all five coins are heads) follows a geometric distribution (as detailed on page 117) with  $p = \frac{1}{32}$  and  $q = 1 - p = \frac{31}{32}$ . Hence:

$$\Pr(N \geq 20) = 1 - \Pr(N \leq 19) = 1 - (1 - q^{19}) = q^{19} = \left(\frac{31}{32}\right)^{19} = 0.5470$$

**Solution 5.13**

The probability that there are at least 4 injuries in a game is  $0.09 + 0.04 = 0.13$ .

Let  $X$  be the number of games before the first game in which there are at least 4 injuries. Then  $X$  follows a geometric distribution with  $p = 0.13$  and  $q = 1 - p = 0.87$ . Hence:

$$E[X] = \frac{q}{p} = \frac{0.87}{0.13} = 6.6923$$

**Solution 5.14**

Let  $X$  be the random number of trucks with engine problems in this sample. If we define a truck with engine problems to be Type 1 and a truck without engine problems to be Type 2, then  $X$  follows a hypergeometric distribution with parameters:

$$m = 20 \quad m_1 = 6 \quad m_2 = 14 \quad n = 4$$

The probability that exactly 2 of the 4 trucks tested have engine problems is:

$$\Pr(X = 2) = \frac{{}_6C_2 {}_{14}C_2}{{}_{20}C_4} = 0.2817$$

**Solution 5.15**

Let  $X$  be the random number of hearts in the sample. If we define a heart to be Type 1 and all other suits to be Type 2, then  $X$  follows a hypergeometric distribution with parameters:

$$m = 52 \quad m_1 = 13 \quad m_2 = 39 \quad n = 5$$

Then we have:

$$(i) \quad E[X] = \frac{nm_1}{m} = \frac{5 \times 13}{52} = 1.25$$

$$(ii) \quad \text{var}(X) = n \left(\frac{m_1}{m}\right) \left(\frac{m_2}{m}\right) \left(\frac{m-n}{m-1}\right) = 5 \times \frac{13}{52} \times \frac{39}{52} \times \frac{47}{51} = 0.8640$$

$$(iii) \quad \Pr(X = 0) = \frac{{}_{13}C_0 {}_{39}C_5}{{}_{52}C_5} = 0.2215$$

**Solution 5.16**

Let  $X$  be the number of accidents in the next year. Then  $X$  follows a Poisson distribution with  $\lambda = 5$ , and the required probability is:

$$\Pr(X = 3) = \frac{e^{-\lambda} \lambda^3}{3!} = \frac{e^{-5} 5^3}{3!} = 0.1404$$

**Solution 5.17**

Let  $X$  be the number of hurricanes in a particular year. Then  $X$  follows a Poisson distribution with  $\lambda = 2.8$ , and we have:

$$\Pr(X = 0) = e^{-\lambda} = e^{-2.8} = 0.06081$$

$$\begin{aligned} \Pr(X \geq 3) &= 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2) \\ &= 1 - e^{-2.8} - \frac{e^{-2.8} 2.8}{1!} - \frac{e^{-2.8} 2.8^2}{2!} \\ &= 1 - 0.06081 - 0.17027 - 0.23838 = 0.53054 \end{aligned}$$

Hence, the required probability is:

$$\Pr(X \geq 3 | X \geq 1) = \frac{\Pr(X \geq 3 \cap X \geq 1)}{\Pr(X \geq 1)} = \frac{\Pr(X \geq 3)}{1 - \Pr(X = 0)} = \frac{0.53054}{1 - 0.06081} = 0.5649$$

**Solution 5.18**

For a Poisson distribution we have:

$$\Pr(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

Hence:

$$3 = \frac{\Pr(X = 2)}{\Pr(X = 4)} = \frac{e^{-\lambda} \lambda^2 / 2!}{e^{-\lambda} \lambda^4 / 4!} = \frac{12}{\lambda^2} \Rightarrow \lambda = 2$$

$$\Rightarrow \text{var}(X) = \lambda = 2$$

**Solution 5.19**

Let  $N$  be the number of consecutive days of rain beginning April 1.

Then  $N$  follows a Poisson distribution with  $\lambda = 0.6$ , so:

$$\Pr(N = n) = \frac{e^{-0.6} 0.6^n}{n!} \quad \text{for } n = 0, 1, \dots$$

The amount paid,  $X$ , is zero with probability:

$$\Pr(N = 0) = \frac{e^{-0.6} 0.6^0}{0!} = 0.54881$$

The amount paid,  $X$ , is 1,000 with probability:

$$\Pr(N = 1) = \frac{e^{-0.6} 0.6^1}{1!} = 0.32929$$

The amount paid,  $X$ , is 2,000 with probability:

$$\Pr(N \geq 2) = 1 - \Pr(N = 0) - \Pr(N = 1) = 0.12190$$

Thus, we have:

$$E[X] = 0 \times 0.54881 + 1,000 \times 0.32929 + 2,000 \times 0.1219 = 573.1$$

$$E[X^2] = 0^2 \times 0.54881 + 1,000^2 \times 0.32929 + 2,000^2 \times 0.1219 = 816,900$$

$$\text{var}(X) = E[X^2] - (E[X])^2 = 488,456$$

So the standard deviation is  $\sqrt{\text{var}(X)} = 699$ .

**Solution 5.20**

The sample mean is:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

Let's also define the sample total:

$$S = X_1 + X_2 + X_3 + X_4 = 4\bar{X}$$

By the additive property,  $S$  follows a Poisson distribution with parameter  $4\lambda$ .

Hence the required probability is:

$$\begin{aligned} \Pr(\bar{X} < 0.5) &= \Pr(S < 2) \\ &= \Pr(S = 0) + \Pr(S = 1) \\ &= e^{-4\lambda} + 4\lambda e^{-4\lambda} \\ &= e^{-4\lambda} (1 + 4\lambda) \end{aligned}$$