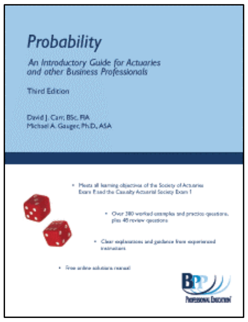


# Probability, Third Edition

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## *Solutions to review questions*

### *Solution 1*

⇨ *From 2010, this question is no longer covered by the Exam P syllabus.*

### *Solution 2*

Answer: **C**

⇨ *This topic is covered in Chapter 1.*

⇨ *The definition of exhaustive events is in Section 1.2.*

Since they are exhaustive events, we have:

$$\Pr(A) + \Pr(A' \cap B) = 1$$

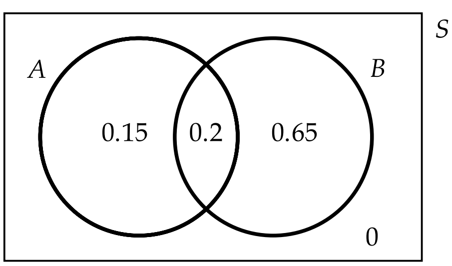
But:

$$\Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B) = 0.65$$

So:

$$\Pr(A) = 0.35$$

*Alternatively, we could show this situation on a Venn diagram and calculate the probability from there:*



## *Solution 3*

Answer: **A**

⇨ *This topic is covered in Chapter 9.*

⇨ *We are using the method of distribution functions here. Examples of this method can be found in Section 9.2.*

Note that $Y = 1.07X$. Consider $F(y) = \Pr(Y < y)$:

$$F(y) = \Pr(Y < y) = \Pr(1.07X < y) = \Pr\left(X < \frac{y}{1.07}\right)$$

We can calculate this by integrating the probability density function over the appropriate range:

$$\int_0^{\frac{y}{1.07}} \frac{32}{(x+4)^3}\,dx = \left(\frac{32(x+4)^{-2}}{-2}\right)\Bigg|_0^{\frac{y}{1.07}} = -16\left(\frac{y}{1.07}+4\right)^{-2} + 1 = 1 - \frac{16\times 1.07^2}{(y+4\times 1.07)^2}$$

Hence:

$$F(y) = 1 - \frac{18.3184}{(y+4.28)^2}$$

We need to differentiate this to get the probability density function:

$$f(y) = F'(y) = \frac{2\times 18.3184}{(y+4.28)^3} = \frac{36.6368}{(y+4.28)^3}$$

## *Solution 4*

Answer: **D**

⇨ *This topic is covered in Chapter 2.*

⇨ *If the order of allocation matters we need to use permutations. This can be found in Section 2.2.*

We need to allocate the difficult case first. It can be allocated to any of the three experts. There are three ways of doing this.

Now we have three claims to be allocated to the remaining nine people. This can be done in the following number of ways:

$${}_9P_3 = \frac{9!}{6!} = 504$$

The total number of ways that the claims can be allocated is:

$$3\times 504 = 1{,}512$$

## *Solution 5*

Answer: **B**

⇨ *This topic is covered in Chapters 4 and 6.*

⇨ *The definition of the interquartile range is in Section 4.4.*

We need $F(x)$:

$$F(x)=\int_0^x \frac{1}{32}t^7 e^{-\frac{t^8}{256}}\,dt$$

Now $\int g'(x)e^{g(x)}dx = e^{g(x)} + \text{ constant}$, so:

$$F(x) = -e^{-\frac{t^8}{256}}\Bigg|_0^x = 1-e^{-\frac{x^8}{256}}$$

The lower quartile, $L$, is such that:

$$0.25 = 1-e^{-\frac{L^8}{256}} \quad\Rightarrow\quad e^{-\frac{L^8}{256}} = 0.75 \quad\Rightarrow\quad L = 1.71157$$

The upper quartile, $U$, is such that:

$$0.75 = 1-e^{-\frac{U^8}{256}} \quad\Rightarrow\quad e^{-\frac{U^8}{256}} = 0.25 \quad\Rightarrow\quad U = 2.08335$$

So the interquartile range is $2.08335-1.71157=0.3718$.

## *Solution 6*

Answer: **B**

⇨ *This topic is covered in Chapters 4 and 6.*

⇨ *We are told that the probability density function given in the question is that of a gamma distribution. It is important to remember the formulas for the mean and variance of the gamma distribution since they are difficult to derive in the exam. They are given in Section 6.4. The formulas are:*

$$E[X]=\alpha\theta$$
$$\text{var}[X]=\alpha\theta^2$$

The gamma distribution has parameters $\alpha=3$ and $\theta=2$, so the mean is $3\times 2=6$ and the variance is $3\times 2^2=12$. The value "mean plus 2 standard deviations" is:

$$6+2\sqrt{12}=12.9282$$

We will call this $a$ for convenience. We want $\Pr(X>a)$:

$$\Pr(X>a)=\int_a^\infty \frac{1}{16}x^2e^{-\frac{1}{2}x}\,dx$$

Integrating this by parts using $u=\frac{1}{16}x^2$, we get:

$$\Pr(X>a)=\left(-\frac{1}{8}x^2e^{-\frac{1}{2}x}\right)\Bigg|_a^\infty+\int_a^\infty \frac{1}{4}xe^{-\frac{1}{2}x}\,dx$$

Integrating this by parts again, but now using $u=\frac{1}{4}x$, we get:

$$\begin{aligned}\Pr(X>a)&=\left(-\frac{1}{8}x^2e^{-\frac{1}{2}x}\right)\Bigg|_a^\infty+\left(-\frac{1}{2}xe^{-\frac{1}{2}x}\right)\Bigg|_a^\infty+\int_a^\infty \frac{1}{2}e^{-\frac{1}{2}x}\,dx\\&=\left(-\frac{1}{8}x^2e^{-\frac{1}{2}x}\right)\Bigg|_a^\infty+\left(-\frac{1}{2}xe^{-\frac{1}{2}x}\right)\Bigg|_a^\infty+\left(-e^{-\frac{1}{2}x}\right)\Bigg|_a^\infty\\&=\frac{1}{8}a^2e^{-\frac{1}{2}a}+\frac{1}{2}ae^{-\frac{1}{2}a}+e^{-\frac{1}{2}a}\end{aligned}$$

Substituting the value of $a$ into this expression, we get:

$$\Pr(X>a)=0.0442$$

## *Solution 7*

Answer: **E**

⇨ *This topic is covered in Chapter 4.*

⇨ *The definition of a moment generating function is given in Section 4.6. We also need the fact that the moment generating function of the sum of independent random variables is the product of the individual moment generating functions. This is also given in Section 4.6.*

The total pay out is:

$$S = 100N_1 + 200N_2 + 300N_3$$

where $N_i$ is the number of claims for a Type $i$ policy.

Working from the definition of the moment generating function:

$$\begin{aligned} M_S(t) = E[e^{tS}] &= E[\exp(t(100N_1 + 200N_2 + 300N_3))] \\ &= E[\exp(100tN_1)\exp(200tN_2)\exp(300tN_3)] \end{aligned}$$

Since the policies are independent:

$$\begin{aligned} M_S(t) &= E[\exp(100tN_1)]E[\exp(200tN_2)]E[\exp(300tN_3)] \\ &= M_{N_1}(100t)M_{N_2}(200t)M_{N_3}(300t) \\ &= \exp[2(e^{100t} - 1)]\exp[2(e^{200t} - 1)]\exp[3(e^{300t} - 1)] \\ &= \exp[2e^{100t} + 2e^{200t} + 3e^{300t} - 7] \end{aligned}$$

## *Solution 8*

Answer: **B**

⇨ *This topic is covered in Chapter 9.*

The mean of the claim amounts without the deductible is:

$$\frac{1}{0.005} = 200$$

⇨ *If you cannot remember the formula for the mean of the exponential distribution, then you would have to integrate.*

The amount paid out with the deductible is $Y$, where:

$$Y = \begin{cases} 0 & X < 75 \\ X - 75 & X \geq 75 \end{cases}$$

⇨ *This formula is detailed in Section 9.3.*

We need $E[Y]$:

$$E[Y] = \int_{75}^{\infty} (x-75)0.005e^{-0.005x} \ dx$$

Integrating this by parts with $u = x - 75$, we get:

$$E[Y] = \left(-(x-75)e^{-0.005x}\right)\Big|_{75}^{\infty} + \int_{75}^{\infty} e^{-0.005x} \ dx$$

$$= 0 + \left(-\frac{1}{0.005}e^{-0.005x}\right)\Big|_{75}^{\infty} = \frac{e^{-0.005 \times 75}}{0.005} = 137.46$$

So the actual reduction is 62.54, and the percentage reduction is:

$$\frac{62.54}{200} \times 100 = 31.3\%$$

## *Solution 9*

Answer: **C**

⇨ *This topic is covered in Chapters 4 and 6.*

Consider $E[X]$:

$$E[X] = \int_0^\infty \alpha 4^\alpha x(x+4)^{-\alpha-1}\ dx$$

Integrating this by parts with $u = \alpha 4^\alpha x$, we get:

$$E[X] = \left(-4^\alpha x(x+4)^{-\alpha}\right)\Big|_0^\infty + \int_0^\infty 4^\alpha (x+4)^{-\alpha}\ dx$$

$$= 0 + \left(\frac{4^\alpha (x+4)^{-\alpha+1}}{-\alpha+1}\right)\Bigg|_0^\infty = -\frac{4^\alpha 4^{-\alpha+1}}{-\alpha+1} = \frac{4}{\alpha-1}$$

*Alternatively, you could have integrated this by substitution.*

But we are told that $E[X] = \frac{4}{3}$, so:

$$\frac{4}{\alpha-1} = \frac{4}{3} \quad \Rightarrow \quad \alpha = 4$$

## *Solution 10*

Answer: **A**

⇨ *This topic is covered in Chapters 4 and 6.*

⇨ *The formula for skewness is in Section 4.5.*

The skewness is:

$$\frac{E\left[\left(X-\frac{4}{3}\right)^3\right]}{\sigma^3}$$

where $\sigma^2 = \text{var}[X]$.

To calculate the variance, we need $E[X^2]$:

$$E[X^2] = \int_0^\infty 4^5 x^2 (x+4)^{-5}\ dx$$

Integrating this by parts with $u = 4^5 x^2$, we get:

$$E[X^2] = \left( \frac{4^5 x^2 (x+4)^{-4}}{-4} \right) \Bigg|_0^\infty + \int_0^\infty 2 \times 4^4 x (x+4)^{-4} \; dx = 0 + \int_0^\infty 2 \times 4^4 x (x+4)^{-4} \; dx$$

Integrating this by parts again, but with $u = 2 \times 4^4 x$, we get:

$$E[X^2] = \left( \frac{2 \times 4^4 x (x+4)^{-3}}{-3} \right) \Bigg|_0^\infty + \int_0^\infty \frac{2 \times 4^4}{3} (x+4)^{-3} \; dx$$

$$= 0 + \left( \frac{2 \times 4^4 (x+4)^{-2}}{3 \times -2} \right) \Bigg|_0^\infty = \frac{4^4}{3} 4^{-2} = \frac{16}{3}$$

*Alternatively, this integration could have been done by substitution, with* $t = x + 4$.

So:

$$\sigma^2 = E[X^2] - E^2[X] = \frac{16}{3} - \left( \frac{4}{3} \right)^2 = \frac{32}{9}$$

We now want $E\left[ \left( X - \frac{4}{3} \right)^3 \right]$:

$$E\left[ \left( X - \frac{4}{3} \right)^3 \right] = \int_0^\infty 4^5 \left( x - \frac{4}{3} \right)^3 (x+4)^{-5} \; dx$$

Rather than integrating by parts, we will integrate this using substitution, with $u = x + 4$:

$$E\left[ \left( X - \frac{4}{3} \right)^3 \right] = \int_4^\infty 4^5 \left( u - \frac{16}{3} \right)^3 u^{-5} \; du = 4^5 \int_4^\infty \left( u^3 - 3u^2 \frac{16}{3} + 3u \left( \frac{16}{3} \right)^2 - \left( \frac{16}{3} \right)^3 \right) u^{-5} \; du$$

$$= 4^5 \int_4^\infty \left( u^{-2} - 16u^{-3} + \frac{256}{3} u^{-4} - \frac{4{,}096}{27} u^{-5} \right) du$$

$$= 4^5 \left( \frac{u^{-1}}{-1} - \frac{16u^{-2}}{-2} + \frac{256}{3} \frac{u^{-3}}{-3} - \frac{4{,}096}{27} \frac{u^{-4}}{-4} \right) \Bigg|_4^\infty$$

$$= 47.4074$$

So the skewness is:

$$\frac{47.4074}{\left( {}^{32}\!/\!_{9} \right)^{1.5}} = 7.071$$

## *Solution 11*

Answer: **C**

⇨ *This topic is covered in Chapter 6.*

⇨ *The memoryless property of the exponential distribution is derived in Section 6.3.*

We want $\Pr(X>12\,|\,X>7)$. Since the exponential distribution has the memoryless property:

$$\Pr(X>12\,|\,X>7)=\Pr(X>5)$$

The probability density function of this exponential distribution is:

$$\frac{1}{8}e^{-\frac{x}{8}} \quad x>0$$

So:

$$\Pr(X>5)=\int_5^\infty \frac{1}{8}e^{-\frac{x}{8}}\,dx=\left(-e^{-\frac{x}{8}}\right)\Big|_5^\infty=e^{-\frac{5}{8}}=0.535$$

*Alternatively, if you had not remembered the memoryless property of the exponential distribution, you would have proceeded as follows:*

$$\Pr(X>12\,|\,X>7)=\frac{\Pr(X>12\cap X>7)}{\Pr(X>7)}=\frac{\Pr(X>12)}{\Pr(X>7)}=\frac{e^{-\frac{12}{8}}}{e^{-\frac{7}{8}}}=e^{-\frac{5}{8}}$$

## *Solution 12*

Answer: **B**

⇨ *This topic is covered in Chapter 4.*

⇨ *The formula for the median of a random variable is in Section 4.3. The pdf given is not from a standard distribution, so you will need to integrate the pdf to find the median.*

We require $F(m)=0.5$ where $m$ is the median value. To calculate the cumulative distribution function, we need to integrate the probability density function:

$$F(x)=\int_0^x 288t^3(t^4+6)^{-3}\,dt=\left(\frac{288}{4}\frac{(t^4+6)^{-2}}{-2}\right)\Big|_0^x=1-\frac{36}{(x^4+6)^2}$$

Since $F(m)=0.5$, we have:

$$1-\frac{36}{(m^4+6)^2}=0.5 \quad\Rightarrow\quad m^4+6=\sqrt{72} \quad\Rightarrow\quad m=1.26$$

## *Solution 13*

Answer: **B**

⇨ *This topic is covered in Chapter 4.*

⇨ *The formula for the mode of a random variable is in Section 4.3. The formula for calculating percentiles is in Section 4.4.*

To calculate the mode, we need to maximize the probability density function, $f(x) = 288x^3(x^4+6)^{-3}$ :

$$\begin{aligned} f'(x) &= 288x^3 \times -3(x^4+6)^{-4} \times 4x^3 + (x^4+6)^{-3} \times 288 \times 3x^2 \\ &= 288x^2(x^4+6)^{-4}\{-12x^4 + 3(x^4+6)\} \\ &= 864x^2(x^4+6)^{-4}\{6-3x^4\} \end{aligned}$$

*Alternatively, you could have maximized* $\ln f(x)$ *to get the same answer.*

This is equal to zero when:

$$6 - 3x^4 = 0 \quad \Rightarrow \quad x = 1.1892$$

The 80th percentile, $p$, is given by $F(p) = 0.8$. Using the information from the previous solution:

$$1 - \frac{36}{(p^4+6)^2} = 0.8 \quad \Rightarrow \quad p^4 + 6 = \sqrt{180} \quad \Rightarrow \quad p = 1.6502$$

The difference between these values is 0.461.

## Solution 14

Answer: **B**

⇨ *This topic is covered in Chapter 8. Joint continuous distributions are covered in Section 8.2. Calculating the mean of a continuous random variable is covered in Section 8.3.*

We need to calculate the value of the constant $a$. Since the total probability is 1:

$$\begin{aligned}
1 = \int_{y=0}^{12.5}\int_{x=0}^{25-2y} a(25-x-2y)\ dx\ dy &= a\int_{y=0}^{12.5}\left(25x-\frac{1}{2}x^2-2xy\right)\Bigg|_{x=0}^{25-2y}\ dy \\
&= a\int_{y=0}^{12.5} 25(25-2y)-\frac{1}{2}(25-2y)^2-2(25-2y)y\ dy \\
&= a\left(25\frac{(25-2y)^2}{-4}-\frac{(25-2y)^3}{-12}-25y^2+\frac{4}{3}y^3\right)\Bigg|_{y=0}^{12.5} \\
&= a\left\{-3,906.25+\frac{7,812.5}{3}-\frac{15,625}{-4}+\frac{15,625}{-12}\right\}
\end{aligned}$$

Solving this we get $a=\frac{12}{15,625}$. To calculate $E[Y]$, we need $f(y)$:

$$\begin{aligned}
f(y) &= \int_{x=0}^{25-2y} a(25-x-2y)\ dx \\
&= \left(a(25x-\frac{1}{2}x^2-2yx)\right)\Bigg|_{x=0}^{25-2y} \\
&= a\left[25(25-2y)-\frac{1}{2}(25-2y)^2-50y+4y^2\right] \\
&= a\left\{2y^2-50y+312.5\right\}
\end{aligned}$$

So $E[Y]$ is:

$$\begin{aligned}
\int_{y=0}^{12.5} yf(y)\ dy = a\int_{y=0}^{12.5} 2y^3-50y^2+312.5y\ dy &= a\left(\frac{1}{2}y^4-\frac{50}{3}y^3+156.25y^2\right)\Bigg|_{y=0}^{12.5} \\
&= a\left(\frac{12.5^4}{2}-\frac{50\times 12.5^3}{3}+156.25\times 12.5^2\right)=3.125
\end{aligned}$$

## *Solution 15*

Answer: **D**

⇨ *This topic is covered in Chapter 8.*

⇨ *Joint continuous distributions are covered in Section 8.2. Calculating the conditional probability density function is in this section and calculating conditional moments is in Section 8.4.*

The conditional probability density function of $X|Y=5$ is:

$$f_X(x|Y=5)=\frac{f_{X,Y}(x,5)}{f_Y(5)}=\frac{a(25-x-10)}{a(2\times 25-50\times 5+312.5)}=\frac{15-x}{112.5}$$

Because of the value of $Y$, we now have $0<x<15$.

We need $E[X|Y=5]$ and $E[X^2|Y=5]$:

$$E[X|Y=5]=\int_0^{15} x\frac{15-x}{112.5}\,dx=\frac{1}{112.5}\left(\frac{15x^2}{2}-\frac{x^3}{3}\right)\Bigg|_0^{15}=\frac{1}{112.5}\left[\frac{15^3}{2}-\frac{15^3}{3}\right]=5$$

$$E[X^2|Y=5]=\int_0^{15} x^2\frac{15-x}{112.5}\,dx=\frac{1}{112.5}\left(\frac{15x^3}{3}-\frac{x^4}{4}\right)\Bigg|_0^{15}=\frac{1}{112.5}\left[\frac{15^4}{3}-\frac{15^4}{4}\right]=37.5$$

Finally, we have:

$$\text{var}[X|Y=5]=37.5-5^2=12.5$$

## *Solution 16*

Answer: **C**

⇨ *This topic is covered in Chapter 1.*

Let $x$ be the number of green discs, so that we have $2x$ blue discs.

We can get three discs of the same color by getting all blue or all green discs. The probability of this occurring is:

$$\frac{2x(2x-1)(2x-2)}{(3x+1)(3x)(3x-1)}+\frac{x(x-1)(x-2)}{(3x+1)(3x)(3x-1)}$$

But we know that this is $\frac{7}{40}$, so:

$$\frac{2x(2x-1)(2x-2)}{(3x+1)(3x)(3x-1)}+\frac{x(x-1)(x-2)}{(3x+1)(3x)(3x-1)}=\frac{7}{40}$$

Solving this:

$$\begin{aligned}
&40\left\{2x(4x^2-6x+2)+x(x^2-3x+2)\right\}=7\times 3x(9x^2-1)\\
\Rightarrow\quad &40\left\{8x^3-12x^2+4x+x^3-3x^2+2x\right\}=189x^3-21x\\
\Rightarrow\quad &171x^3-600x^2+261x=0\\
\Rightarrow\quad &171x^2-600x+261=0
\end{aligned}$$

Solving this equation using the quadratic formula:

$$x=\frac{600\pm\sqrt{600^2-4\times 171\times 261}}{2\times 171}=3\text{ or }0.509$$

We need $x$ to be a whole number, so $x=3$ and the number of blue discs is 6.

*Alternatively, you could have tried each of the solutions in turn until you found the correct probability.*

## *Solution 17*

Answer: **C**

⇨ *This topic is covered in Chapter 3.*

Let the event 'the first two discs are the same color' be $A$. Let the event 'the third disc is green' be $G_3$. We require $\Pr(G_3 \mid A)$:

$$\Pr(G_3 \mid A) = \frac{\Pr(G_3 \cap A)}{\Pr(A)}$$

From the previous question, we know that there are 6 blue discs, 3 green discs and 1 red disc in the bag. For $\Pr(A)$:

$$\Pr(A) = \frac{6}{10} \times \frac{5}{9} + \frac{3}{10} \times \frac{2}{9} = \frac{36}{90} = \frac{2}{5}$$

The event $G_3 \cap A$ can happen in 2 ways; either all three discs are green or the first two are blue and the third is green:

$$\Pr(G_3 \cap A) = \frac{6}{10} \times \frac{5}{9} \times \frac{3}{8} + \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{96}{720} = \frac{2}{15}$$

Finally:

$$\Pr(G_3 \mid A) = \frac{2/15}{2/5} = \frac{1}{3}$$

## *Solution 18*

Answer: **D**

⇨ *This topic is covered in Chapter 6.*

⇨ *The uniform distribution can be found in Section 6.1, where the formulas for the mean and variance of the uniform distribution can also be found.*

We need to calculate $a$ and $b$. For the $U(a,b)$ distribution, we know that:

$$E[X]=\frac{1}{2}(a+b)$$
$$\text{var}[X]=\frac{1}{12}(b-a)^2$$

So:

$$150=\frac{1}{2}(a+b)$$
$$1,875=\frac{1}{12}(b-a)^2$$

From the first equation, $b=300-a$. Substituting this into the second equation:

$$\frac{1}{12}(300-a-a)^2=1,875 \quad\Rightarrow\quad 300-2a=150 \quad\Rightarrow\quad a=75, b=225$$

So the probability density function is $f(x)=\dfrac{1}{225-75}=\dfrac{1}{150}$ for $75<x<225$.

We want $\Pr(X>170\,|\,X>120)$:

$$\Pr(X>170\,|\,X>120)=\frac{\Pr(X>170\cap X>120)}{\Pr(X>120)}=\frac{\Pr(X>170)}{\Pr(X>120)}$$

We will calculate this by integration:

$$\Pr(X>170\,|\,X>120)=\frac{\int_{170}^{225}\frac{1}{150}\,dx}{\int_{120}^{225}\frac{1}{150}\,dx}=\frac{\frac{1}{150}(225-170)}{\frac{1}{150}(225-120)}=0.524$$

## *Solution 19*

Answer: **E**

⇨ *This topic is covered in Chapter 5.*

⇨ *The hypergeometric distribution can be found in Section 5.4. This distribution can be used when you are sampling without replacement from a finite population.*

Let $X$ be the number of red balls in the sample. Then $X$ follows a hypergeometric distribution with parameters $m = 56$, $m_1 = 30$, $m_2 = 26$ and $n = 8$.

$$\Pr(X = 5) = \frac{{}_{30}C_5 \; {}_{26}C_3}{{}_{56}C_8} = 0.2608$$

## *Solution 20*

Answer: **E**

⇨ *This topic is covered in Chapters 1, 4 and 5.*

⇨ *One of the important pieces of information is that the claim amounts and the number of claims is independent, which means that we can multiply the probabilities.*

We need to work out how the total claim can be less than $4,000, and this means considering the number of claims as well as the claim amount.

| Number of claims | Claim 1 | Claim 2 | Claim 3 |
|---|---|---|---|
| 0 | – | – | – |
| 1 | 3 | – | – |
| 1 | 2 | – | – |
| 1 | 1 | – | – |
| 2 | 2 | 1 | – |
| 2 | 1 | 2 | – |
| 2 | 1 | 1 | – |
| 3 | 1 | 1 | 1 |

If the number of claims is $N$, we require $\Pr(N=0)$, $\Pr(N=1)$, $\Pr(N=2)$ and $\Pr(N=3)$:

$$\Pr(N=0)=\frac{e^{-3}3^0}{0!}=0.049787 \qquad \Pr(N=1)=\frac{e^{-3}3^1}{1!}=0.149361$$

$$\Pr(N=2)=\frac{e^{-3}3^2}{2!}=0.224042 \qquad \Pr(N=3)=\frac{e^{-3}3^3}{3!}=0.224042$$

Calculating the individual probabilities:

| Number of claims | Claim 1 | Claim 2 | Claim 3 | Probability |
|---|---|---|---|---|
| 0 | – | – | – | $0.049787$ |
| 1 | 3 | – | – | $0.149361\times0.15=0.022404$ |
| 1 | 2 | – | – | $0.149361\times0.3=0.044808$ |
| 1 | 1 | – | – | $0.149361\times0.4=0.059744$ |
| 2 | 2 | 1 | – | $0.224042\times0.3\times0.4=0.026885$ |
| 2 | 1 | 2 | – | $0.224042\times0.4\times0.3=0.026885$ |
| 2 | 1 | 1 | – | $0.224042\times0.4^2=0.035847$ |
| 3 | 1 | 1 | 1 | $0.224042\times0.4^3=0.014339$ |
| | | | Total | $0.281$ |

## *Solution 21*

Answer: **A**

⇨ *This topic is covered in Chapter 6.*

We need to calculate $a$. Since the total probability must be 1, we know that:

$$1=\int_0^1 ax(1-x)^4\ dx$$

Integrating this by parts with $u=ax$, we get:

$$\begin{aligned}\int_0^1 ax(1-x)^4\ dx &= \left(-\frac{ax(1-x)^5}{5}\right)\Bigg|_0^1+\int_0^1\frac{a}{5}(1-x)^5\ dx\\ &= \left(-\frac{ax(1-x)^5}{5}\right)\Bigg|_0^1+\left(-\frac{a(1-x)^6}{30}\right)\Bigg|_0^1\\ &= 0+0-\left(-\frac{a}{30}\right)=\frac{a}{30}\end{aligned}$$

Since this is equal to 1, we know that $a=30$.

We then need to calculate $E[X]$ and $E[X^2]$. Both of these will be integrated by parts:

$$\begin{aligned}E[X]=\int_0^1 30x^2(1-x)^4\ dx &= \left(-\frac{30x^2(1-x)^5}{5}\right)\Bigg|_0^1+\int_0^1 12x(1-x)^5\ dx\\ &= 0+\left(-2x(1-x)^6\right)\Big|_0^1+\int_0^1 2(1-x)^6\ dx\\ &= 0+\left(-\frac{2}{7}(1-x)^7\right)\Bigg|_0^1=\frac{2}{7}\end{aligned}$$

$$\begin{aligned}E[X^2]=\int_0^1 30x^3(1-x)^4\ dx &= \left(-6x^3(1-x)^5\right)\Big|_0^1+\int_0^1 18x^2(1-x)^5\ dx\\ &= 0+\left(-3x^2(1-x)^6\right)\Big|_0^1+\int_0^1 6x(1-x)^6\ dx\\ &= 0+\left(-\frac{6}{7}x(1-x)^7\right)\Bigg|_0^1+\int_0^1\frac{6}{7}(1-x)^7\ dx\\ &= \left(-\frac{6}{7}\frac{(1-x)^8}{8}\right)\Bigg|_0^1=\frac{3}{28}\end{aligned}$$

*Alternatively, you could have integrated this by substitution.*

Finally, we can get the variance:

$$\text{var}[X] = \frac{3}{28} - \left(\frac{2}{7}\right)^2 = \frac{5}{196}$$

The expression 'mean plus or minus 3 standard deviations' is equal to:

$$\frac{2}{7} \pm 3\sqrt{\frac{5}{196}} = (-0.19344, 0.76487)$$

Since $0 < x < 1$, we require $\Pr(X > 0.76487)$:

$$\Pr(X > 0.76487) = \int_{0.76487}^{1} 30x(1-x)^4 \; dx$$

From previous working:

$$\begin{aligned}\Pr(X > 0.76487) &= \int_{0.76487}^{1} 30x(1-x)^4 \; dx \\ &= \left(-\frac{30x(1-x)^5}{5}\right)\Bigg|_{0.76487}^{1} + \left(-\frac{30(1-x)^6}{30}\right)\Bigg|_{0.76487}^{1} \\ &= 6 \times 0.76487(1-0.76487)^5 + (1-0.76487)^6 = 0.003467\end{aligned}$$

## *Solution 22*

Answer: **E**

⇨ *This topic is covered in Chapter 5.*

⇨ *The formulas for the negative binomial distribution are listed in Section 5.2.*

We know that for the negative binomial distribution:

$$\Pr(X = x) = {}_{r+x-1}C_x \, p^r q^x$$

So we have:

$$\Pr(X = 8) = {}_{r+7}C_8 \, p^r q^8$$
$$\Pr(X = 6) = {}_{r+5}C_6 \, p^r q^6$$

So:

$$\frac{\Pr(X=8)}{\Pr(X=6)} = \frac{{}_{r+7}C_8 \, p^r q^8}{{}_{r+5}C_6 \, p^r q^6} = \frac{\dfrac{(r+7)!}{8!(r-1)!}}{\dfrac{(r+5)!}{6!(r-1)!}} q^2$$
$$= \frac{(r+7)!6!}{(r+5)!8!} q^2$$
$$= \frac{(r+7)(r+6)}{8 \times 7} \times 0.42^2$$

But we know that this is equal to 0.3465, so:

$$\frac{(r+7)(r+6)}{8 \times 7} \times 0.42^2 = 0.3465 \quad \Rightarrow \quad (r+7)(r+6) = 110$$
$$\Rightarrow \quad r^2 + 13r + 42 = 110$$
$$\Rightarrow \quad r^2 + 13r - 68 = 0$$
$$\Rightarrow \quad (r-4)(r+17) = 0$$
$$\Rightarrow \quad r = 4 \text{ or } r = -17$$

Since $r$ must be positive, we have $r = 4$.

Now the annual number of claims for 10 policies, $\sum_{i=1}^{10} X_i$, also has a negative binomial distribution with parameters $r = 4 \times 10 = 40$ and $p = 0.58$. We want $\Pr\left(\sum_{i=1}^{10} X_i = 27\right)$:

$$\Pr\left(\sum_{i=1}^{10} X_i = 27\right) = {}_{66}C_{27} (0.58)^{40} (0.42)^{27} = 0.0568$$

## *Solution 23*

Answer: **C**

⇨ *This topic is covered in Chapter 6.*

⇨ *The question describes a uniform distribution even though it is not named. This is covered in Section 6.1. You can learn the formulas for the uniform distribution, but if you don't, it is very easy to derive the necessary values by integration.*

To have a total probability of 1, we must have:

$$f(x) = \frac{1}{500} = 0.002$$

To calculate the mean:

$$E[X] = \int_{100}^{600} 0.002x \; dx = \left(0.001x^2\right)\Big|_{100}^{600} = 0.001(600^2 - 100^2) = 350$$

⇨ *Alternatively, you could remember that the mean of a symmetrical distribution is the midpoint.*

To calculate the variance, we need $E[X^2]$:

$$E[X^2] = \int_{100}^{600} 0.002x^2 \; dx = \left(0.002\frac{x^3}{3}\right)\Bigg|_{100}^{600} = \frac{430{,}000}{3}$$

So the variance is:

$$\frac{430{,}000}{3} - 350^2 = \frac{62{,}500}{3}$$

and the standard deviation is 144.338.

We want $\Pr(350 - 144.338t < X < 350 + 144.338t) = 0.9$, so:

$$\int_{350-144.338t}^{350+144.338t} 0.002 \; dx = (0.002x)\Big|_{350-144.338t}^{350+144.338t} = 0.002 \times 2t \times 144.338 = 0.9 \quad \Rightarrow \quad t = 1.56$$

## *Solution 24*

Answer: **D**

⇨ *This topic is covered in Chapters 5 and 7.*

⇨ *You need to match the length of time referred to for the Poisson. For example, this question gives us the mean number of claims per day and then asks about the number of claims in a whole year. You must multiply the mean number per day by 365 to get the mean for 2006 (since 2006 is not a leap year). The Poisson distribution is covered in Section 5.5. You then need to remember to apply an approximation. You can find the details of this in Section 7.3.*

The number of claims received in 2006 will have a Poisson distribution with mean:

$$\lambda = 3 \times 365 = 1{,}095$$

We will use an approximation here. Let $X$ be the number of claims, then:

$$X \sim N(1095, 1095)$$

We require $\Pr(X < 1{,}000)$, but we have to apply a continuity correction:

$$\begin{aligned}\Pr(X < 1{,}000) &\approx \Pr(X < 999.5)\\ &= \Pr\left(\frac{X - 1{,}095}{\sqrt{1{,}095}} < \frac{999.5 - 1{,}095}{\sqrt{1{,}095}}\right)\\ &= \Phi(-2.886) = 1 - \Phi(2.886)\\ &= 1 - 0.9981 = 0.0019\end{aligned}$$

## *Solution 25*

Answer: **E**

⇨ *This topic is covered in Chapter 8.*

⇨ *Joint discrete distributions are covered in Section 8.1. Examples of calculating conditional probabilities of joint discrete distributions can also be found in this section.*

We have:

| | | X | |
|---|---|---|---|
| | | 2 | 3 |
| Y | 2 | $a$ | 0.4 |
| | 4 | $b$ | 0.15 |
| | 6 | $c$ | 0.05 |

From the information given to us in the question:

$$c + 0.05 = 0.15 \quad \Rightarrow \quad c = 0.1$$

$$3.4 = 2(a + 0.4) + 4(b + 0.15) + 0.9 \quad \Rightarrow \quad a + 2b = 0.55 \qquad \text{Equation 1}$$

But also, we know that the total probability is 1, so:

$$a + b + 0.7 = 1 \quad \Rightarrow \quad a + b = 0.3 \qquad \text{Equation 2}$$

Subtracting equation 2 from Equation1:

$$b = 0.25$$

From which we get $a = 0.05$.

Finally:

$$\Pr(X = 3 \mid Y = 2) = \frac{\Pr(X = 3 \cap Y = 2)}{\Pr(Y = 2)} = \frac{0.4}{a + 0.4} = 0.889$$

## *Solution 26*

Answer: **A**

⇨ *This topic is covered in Chapter 8.*

⇨ *Joint discrete distributions are covered in Section 8.1. Examples of calculating conditional variances of joint discrete distributions can be found in Section 8.4.*

We want the probability distribution of $Y \mid X = 2$:

| $Y$ | 2 | 4 | 6 |
|---|---|---|---|
| prob $\Pr(Y \mid X = 2)$ | $\frac{0.05}{0.4}$ | $\frac{0.25}{0.4}$ | $\frac{0.1}{0.4}$ |

Then:

$$E[Y \mid X = 2] = 2 \times \frac{0.05}{0.4} + 4 \times \frac{0.25}{0.4} + 6 \times \frac{0.1}{0.4} = 4.25$$
$$E[Y^2 \mid X = 2] = 2^2 \times \frac{0.05}{0.4} + 4^2 \times \frac{0.25}{0.4} + 6^2 \times \frac{0.1}{0.4} = 19.5$$
$$\text{var}[Y \mid X = 2] = 19.5 - 4.25^2 = 1.4375$$

## *Solution 27*

Answer: **B**

⇨ *This topic is covered in Chapter 8.*

⇨ *Joint discrete distributions are covered in Section 8.1. Moments of joint discrete distributions can be found in Section 8.3.*

We have:

$$\begin{aligned}\text{var}[2X+2Y] &= E[(2X+2Y)^2] - E^2[2X+2Y] \\ &= \sum_{x,y}(2x+2y)^2 \Pr(X=x\cap Y=y) - \left\{\sum_{x,y}(2x+2y)\Pr(X=x\cap Y=y)\right\}^2\end{aligned}$$

Calculating the required values:

$$\begin{aligned}\sum_{x,y}(2x+2y)\Pr(X=x\cap Y=y) &= (4+4)\times 0.05+(4+8)\times 0.25+(4+12)\times 0.1 \\ &\quad +(6+4)\times 0.4+(6+8)\times 0.15+(6+12)\times 0.05 \\ &= 0.4+3+1.6+4+2.1+0.9 = 12\end{aligned}$$

$$\begin{aligned}\sum_{x,y}(2x+2y)^2\Pr(X=x\cap Y=y) &= (4+4)^2\times 0.05+(4+8)^2\times 0.25+(4+12)^2\times 0.1 \\ &\quad +(6+4)^2\times 0.4+(6+8)^2\times 0.15+(6+12)^2\times 0.05 \\ &= 3.2+36+25.6+40+29.4+16.2 = 150.4\end{aligned}$$

So the variance is:

$$\text{var}[2X+2Y] = 150.4 - 12^2 = 6.4$$

## *Solution 28*

Answer: **D**

⇨ *This topic is covered in Chapter 8.*

⇨ *The formula for the correlation coefficient is in Section 8.5.*

The correlation coefficient is:

$$\rho = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{var}[X]\text{var}[Y]}}$$

Calculating the required values:

$$E[X] = 2 \times 0.4 + 3 \times 0.6 = 2.6$$
$$E[X^2] = 2^2 \times 0.4 + 3^2 \times 0.6 = 7$$
$$E[Y^2] = 2^2 \times 0.45 + 4^2 \times 0.4 + 6^2 \times 0.15 = 13.6$$
$$\text{var}[X] = 7 - 2.6^2 = 0.24$$
$$\text{var}[Y] = 13.6 - 3.4^2 = 2.04$$
$$E[XY] = 4 \times 0.05 + 6 \times 0.4 + 8 \times 0.25 + 12 \times 0.15 + 12 \times 0.1 + 18 \times 0.05 = 8.5$$

We were told that $E[Y] = 3.4$.

Substituting these into the expression for the correlation coefficient:

$$\rho = \frac{8.5 - 2.6 \times 3.4}{\sqrt{0.24 \times 2.04}} = -0.486$$

## *Solution 29*

Answer: **C**

⇨ *This topic is covered in Chapter 5.*

⇨ *Because there can be a success or failure (attendance or non-attendance), independent events and a maximum number of successes (15 people could attend the party), we have a binomial distribution. This is covered in Section 5.1. The formula for the probability function of the binomial distribution is also in this section.*

The number of people who will attend the party, $N$, has a binomial distribution with parameters $n = 15$ and $p = 0.83$. We want $\Pr(N > 13)$:

$$\begin{aligned}\Pr(N > 13) &= \Pr(N = 14) + \Pr(N = 15) \\ &= {}_{15}C_{14}(0.83)^{14}(0.17) + {}_{15}C_{15}(0.83)^{15} \\ &= 0.249\end{aligned}$$

## *Solution 30*

⇨ *From 2010, this question is no longer covered by the Exam P syllabus.*

## *Solution 31*

Answer: **E**

⇨ *This topic is covered in Chapter 4.*

⇨ *The formula for the variance of $aX+b$ (namely $\text{var}[aX+b]=a^2\,\text{var}[X]$) can be found in Section 4.4.*

If the charge is $C$ and the number of hours is $H$, then we have:

$$C=28H+48$$

We require $\text{var}[C]$:

$$\text{var}[C]=28^2\,\text{var}[H]=28^2\times 5^2=19{,}600$$

## *Solution 32*

Answer: **C**

⇨ *This topic is covered in Chapter 5.*

⇨ *You need to match the length of time referred to for the Poisson. For example, this question gives us the mean number of calls per day and then asks about the number of calls in a week. You must multiply the mean number per day by 5 to get the mean for a working week. The Poisson distribution is covered in Section 5.5, and the formula for the probability function can also be found in this section.*

The number of telephone calls per working week has a Poisson distribution with parameter $5\lambda$.

We are told that $\Pr(X=20)=1.72\Pr(X=15)$, where:

$$\Pr(X=15)=\frac{e^{-5\lambda}(5\lambda)^{15}}{15!}$$

$$\Pr(X=20)=\frac{e^{-5\lambda}(5\lambda)^{20}}{20!}$$

So:

$$\frac{e^{-5\lambda}(5\lambda)^{20}}{20!}=1.72\frac{e^{-5\lambda}(5\lambda)^{15}}{15!}$$

$$\Rightarrow\quad \frac{20!\times 1.72}{15!}=(5\lambda)^5 \quad\Rightarrow\quad \lambda=4$$

We want $\Pr(Y=5)$, where $Y$ is the number of telephone calls in a day. $Y$ has a Poisson distribution with parameter $\lambda=4$:

$$\Pr(Y=5)=\frac{e^{-4}4^5}{5!}=0.156$$

## *Solution 33*

Answer: B

⇨ *This topic is covered in Chapter 4.*

⇨ *The formula for the mean of a random variable is in Section 4.3. The formula for the variance of a random variable is in Section 4.4.*

Let the distribution be as follows:

| $X$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Probability | $a$ | $b$ | $c$ | $d$ | $e$ |

Using the information from the question:

$$a+2b+3c+4d+5e=3 \qquad \text{Equation 1}$$

$$a+4b+9c+16d+25e-9=\frac{28}{11} \qquad \text{Equation 2}$$

$$d+e=\frac{5}{11} \qquad \text{Equation 3}$$

$$a+b=\frac{5}{11} \qquad \text{Equation 4}$$

But also, the sum of the probabilities is 1, so:

$$a+b+c+d+e=1 \qquad \text{Equation 5}$$

Using Equations 3, 4 and 5:

$$c=\frac{1}{11}$$

Substituting this and Equations 3, 4 into Equation 1:

$$1\times\frac{5}{11}+b+\frac{3}{11}+4\times\frac{5}{11}+e=3 \quad\Rightarrow\quad b+e=\frac{5}{11} \qquad \text{Equation 6}$$

Substituting the value of $c$ and Equations 3, 4 into Equation 2:

$$1\times\frac{5}{11}+3b+\frac{9}{11}+16\times\frac{5}{11}+9e-9=\frac{28}{11} \quad\Rightarrow\quad 3b+9e=3 \qquad \text{Equation 7}$$

Equation 7 minus $3\times$Equation 6 gives:

$$6e=\frac{33}{11}-\frac{15}{11}=\frac{18}{11} \quad\Rightarrow\quad e=\frac{3}{11}$$

From this we can get $d=\dfrac{2}{11}$.

Finally:

$$\Pr(X=4\,|\,X>2)=\frac{\Pr(X=4\cap X>2)}{\Pr(X>2)}=\frac{\Pr(X=4)}{\Pr(X>2)}=\frac{d}{c+d+e}=\frac{\frac{2}{11}}{\frac{1}{11}+\frac{2}{11}+\frac{3}{11}}=\frac{1}{3}$$

## *Solution 34*

Answer: **A**

⇨ *This topic is covered in Chapters 5 and 7.*

⇨ *The binomial distribution is covered in Section 5.1. You then need to remember to apply an approximation. You can find the details of this in Section 7.3.*

The number of policyholders with the gene can be modeled by a binomial distribution with parameters $n = 450$ and $p = 20\%$ .

This can be approximated by a normal distribution with parameters:

$$\mu = 450 \times 0.2 = 90$$
$$\sigma^2 = 450 \times 0.2 \times 0.8 = 72$$

We want $\Pr(X > 100)$ , which with continuity correction becomes $\Pr(X > 100.5)$ :

$$\begin{aligned}\Pr(X > 100.5) &= 1 - \Pr\left(\frac{X-90}{\sqrt{72}} < \frac{100.5-90}{\sqrt{72}}\right)\\ &= 1 - \Phi(1.237)\\ &= 1 - 0.892 = 0.108\end{aligned}$$

## *Solution 35*

Answer: **B**

⇨ *This topic is covered in Chapters 4 and 7.*

⇨ *The mean and variance of $aX + b$ are covered in Sections 4.3 and 4.4.*

If the charge is $C$ and the number of hours worked is $H$ then:

$$C = 50H + 60$$

We then know that:

$$E[C] = 50E[H] + 60 = 360$$
$$\text{var}[C] = 50^2 \text{ var}[H] = 10,000$$

So that $C \sim N(360, 10000)$ . We want $\Pr(C > 400)$ :

$$\Pr(C > 400) = \Pr\left(\frac{C-360}{\sqrt{10,000}} > \frac{400-360}{\sqrt{10,000}}\right) = 1 - \Phi(0.4) = 1 - 0.6554 = 0.3446$$

## *Solution 36*

Answer: **C**

⇨ *This topic is covered in Chapter 5.*

⇨ *The geometric distribution is covered in Section 5.3.*

The number of failures, $X$, before she passes an exam has a geometric distribution with parameter $p = 0.6$, so that $P(X = x) = 0.6 \times 0.4^x$ ($x = 0, 1, 2, \dots$).

$$\begin{aligned}\Pr(X \geq 4) &= 1 - (\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)) \\ &= 1 - (0.6 + 0.4 \times 0.6 + 0.4^2 \times 0.6 + 0.4^3 \times 0.6) \\ &= 1 - 0.9744 = 0.0256\end{aligned}$$

## *Solution 37*

Answer: **B**

⇨ *This topic is covered in Chapters 4 and 5.*

⇨ *The Poisson distribution can be found in Section 5.5, moment generating functions in Section 4.6 and skewness in Section 4.5.*

For a Poisson distribution:

$$\frac{\Pr(N = 3)}{\Pr(N = 2)} = \frac{e^{-\lambda}\lambda^3 / 3!}{e^{-\lambda}\lambda^2 / 2!} = \frac{\lambda}{3}$$

But we are told that this is 2, so $\lambda = 6$.

We have $M_N(t) = e^{6(e^t - 1)}$, so that $R_N(t) = \ln M_N(t) = 6(e^t - 1)$.

We know that $E[(X - \mu)^3] = R_N'''(0)$.

$$\begin{aligned}R_N'(t) &= 6e^t \\ R_N''(t) &= 6e^t \\ R_N'''(t) &= 6e^t\end{aligned}$$

So $R_N'''(0) = 6$. Since the mean and variance of the Poisson distribution are $\lambda$, the skewness is:

$$\frac{E[(X - \lambda)^3]}{(\sqrt{\lambda})^3}$$

Substituting in the numerical values:

$$\frac{E[(X - \lambda)^3]}{(\sqrt{\lambda})^3} = \frac{6}{6^{1.5}} = 0.408$$

## *Solution 38*

Answer: **D**

⇨ *This topic is covered in Chapter 3.*

⇨ *We will use Bayes' Theorem, which can be found in Section 3.6.*

Using the obvious notation, from the question, we know the following:

$$\Pr(S\,|\,H)=0.6 \qquad \Pr(S\,|\,L)=0.7 \qquad \Pr(S\,|\,E)=0.55$$
$$\Pr(L)=0.28 \qquad \Pr(E)=0.05 \qquad \Pr(H)=0.67$$

We require $\Pr(L\,|\,S)$:

$$\begin{aligned}\Pr(L\,|\,S)&=\frac{\Pr(L\cap S)}{\Pr(S)}\\&=\frac{\Pr(S\,|\,L)P(L)}{\Pr(S\,|\,H)P(H)+\Pr(S\,|\,L)P(L)+\Pr(S\,|\,E)P(E)}\\&=\frac{0.7\times 0.28}{0.6\times 0.67+0.7\times 0.28+0.55\times 0.05}=0.313\end{aligned}$$

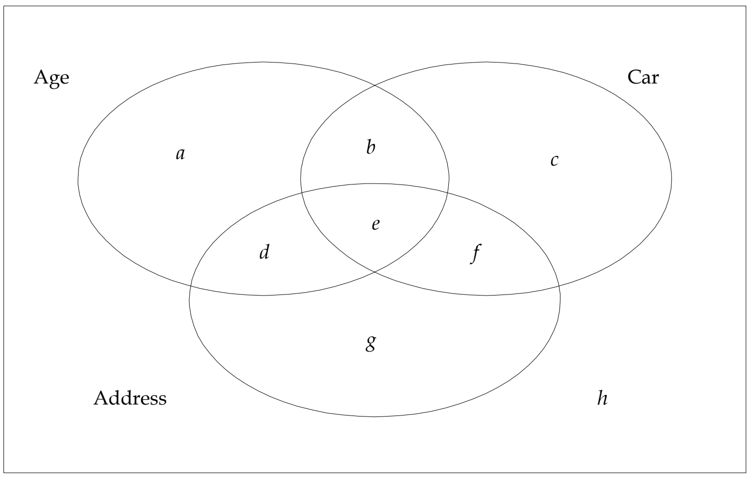
## *Solution 39*

Answer: **A**

⇨ *This topic is covered in Chapter 1.*

⇨ *We will use Venn diagrams, which can be found in Section 1.2.*

The following Venn diagram represents the situation:




We want $e$ here.

From the question, we know the following:

(i) $a+b+d+e=28$
(ii) $d+e+f+g=53$
(iii) $d=6$
(iv) $c=20$
(v) $h=15$

*We don't actually need the last two pieces of information to answer this question.*

Which means that:

(i) $\Rightarrow \quad a+b+e=22$
(ii) $\Rightarrow \quad f+g=47-e$

But $a+b+c+d+e+f+g+h=100$, so substituting in the values of $d$, $c$ and $h$, we have:

$$a+b+e+f+g=59$$

Combining the last three equations, we have:

$$22+47-e=59 \quad \Rightarrow \quad e=10$$

## *Solution 40*

Answer: **C**

⇨ *This topic is covered in Chapter 1.*

We want $b+d+f$ here.

Using the information from the previous question and the rest of the information from the question:

$$b+e=15 \quad \Rightarrow \quad b=5$$
$$e+f=17 \quad \Rightarrow \quad f=7$$

Those classified as high risk under exactly two categories is:

$$b+d+f=5+6+7=18$$

## *Solution 41*

Answer: **A**

⇨ *This topic is covered in Chapter 4.*

⇨ *Generating functions can be found in Section 4.6. The formula for the variance can also be found in this section.*

The variance is:

$$\text{var}[X]=M_X''(0)-[M_X'(0)]^2$$

If we expand the expression for the moment generating function, we get:

$$M_X(t)=\frac{1+100t+\frac{100^2t^2}{2!}+\frac{100^3t^3}{3!}+\frac{100^4t^4}{4!}+\cdots-1}{100t}=1+50t+\frac{100^2}{6}t^2+\frac{100^3}{24}t^3+\cdots$$

*We have used a non-standard method here. If we just took the expression in the question for the moment generating function, differentiated it and then tried to substitute in zero, we would end up dividing by zero.*

Differentiating:

$$M_X'(t)=50+\frac{100^2}{6}\times 2t+\frac{100^3}{24}\times 3t^2+\cdots$$
$$M_X''(t)=\frac{100^2}{6}\times 2+\frac{100^3}{24}\times 6t+\cdots$$

So:

$$M_X'(0)=50$$
$$M_X''(0)=\frac{100^2}{6}\times 2=\frac{100^2}{3}$$

Finally:

$$\text{var}[X]=\frac{100^2}{3}-50^2=833$$

## *Solution 42*

Answer: **B**

⇨ *This topic is covered in Chapter 5.*

⇨ *The hypergeometric distribution can be found in Section 5.4.*

Let $X$ be the number of claims that are less than \$1,000. Then $X$ follows a hypergeometric distribution with parameters:

$$m = 50 \quad m_1 = 37 \quad m_2 = 13 \quad n = 15$$

So:

$$\Pr(X = 8) = \frac{{}_{37}C_8 \; {}_{13}C_7}{{}_{50}C_{15}} = 0.0294$$

## *Solution 43*

Answer: **B**

⇨ *This topic is covered in Chapter 9. Details of dealing with a policy limit are in Section 9.3.*

Let $X$ be the loss amount and $Y$ be the amount paid out, so that:

$$Y = \begin{cases} X & X < 5{,}000 \\ 5{,}000 & X \geq 5{,}000 \end{cases}$$

The probability density function of $X$ is:

$$\frac{1}{4{,}000} e^{-\frac{x}{4{,}000}}$$

We have:

$$E[Y] = \int_0^{5{,}000} \frac{x}{4{,}000} e^{-\frac{x}{4{,}000}}\,dx + \int_{5{,}000}^{\infty} \frac{5{,}000}{4{,}000} e^{-\frac{x}{4{,}000}}\,dx$$

We can integrate the first integral by parts, using $u = x$:

$$E[Y] = \left(-xe^{-\frac{x}{4{,}000}}\right)\Big|_0^{5{,}000} + \int_0^{5{,}000} e^{-\frac{x}{4{,}000}}\,dx + \left(-5{,}000e^{-\frac{x}{4{,}000}}\right)\Big|_{5{,}000}^{\infty}$$

$$= \left(-xe^{-\frac{x}{4{,}000}}\right)\Big|_0^{5{,}000} + \left(-4{,}000e^{-\frac{x}{4{,}000}}\right)\Big|_0^{5{,}000} + \left(-5{,}000e^{-\frac{x}{4{,}000}}\right)\Big|_{5{,}000}^{\infty}$$

$$= (-5{,}000e^{-1.25}) - (0) + (-4{,}000e^{-1.25}) - (-4{,}000) + (0) - (-5{,}000e^{-1.25}) = 2{,}854$$

*Alternatively, the second integral could be calculated as follows:*

$$\int_{5{,}000}^{\infty} \frac{5{,}000}{4{,}000} e^{-\frac{x}{4{,}000}}\,dx = 5{,}000 \int_{5{,}000}^{\infty} \frac{1}{4{,}000} e^{-\frac{x}{4{,}000}}\,dx = 5{,}000 \Pr(X > 5{,}000) = 5{,}000e^{-1.25}$$

## *Solution 44*

⇨ *From 2010, this question is no longer covered by the Exam P syllabus.*

## *Solution 45*

Answer: **D**

⇨ *This topic is covered in Chapter 8.*

⇨ *The bivariate normal distribution is in Section 8.6.*

The formula for the given mean is:

$$E[X \mid Y = 175] = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(175 - \mu_Y)$$

Substituting in the given values:

$$225 = 200 + \rho\sqrt{\frac{8}{5}}(175 - 150) \quad \Rightarrow \quad \rho\sqrt{\frac{8}{5}} = 1 \quad \Rightarrow \quad \rho = 0.79$$

However we require the covariance:

$$\operatorname{cov}(X, Y) = \rho\sigma_X\sigma_Y = 500$$

## *Solution 46*

Answer: **C**

⇨ *This topic is covered in Chapter 9, Section 9.5.*

⇨ *Although the Pareto distribution is no longer on the Exam P syllabus, you can answer this question by knowing the formula for the CDF, which you have been given.*

If $X_i$ is the $i$ th claim amount and $Z = max(X_1, \dots, X_5)$, we have:

$$\begin{aligned}\Pr(Z > 30) &= 1 - \Pr(Z \le 30) \\ &= 1 - \Pr\{(X_1 \le 30) \cap (X_2 \le 30) \cap (X_3 \le 30) \cap (X_4 \le 30) \cap (X_5 \le 30)\} \\ &= 1 - \Pr(X_1 \le 30)\Pr(X_2 \le 30)\Pr(X_3 \le 30)\Pr(X_4 \le 30)\Pr(X_5 \le 30)\end{aligned}$$

But for the Pareto distribution:

$$\Pr(X_i \le 30) = F(30) = 1 - \left(\frac{50}{30 + 50}\right)^4 = 1 - \left(\frac{5}{8}\right)^4$$

So:

$$\Pr(Z > 30) = 1 - \left\{1 - \left(\frac{5}{8}\right)^4\right\}^5 = 0.5630$$

## *Solution 47*

Answer: **C**

⇨ *This topic is covered in Chapter 8.*

We need to calculate the value of the constant:

$$\int_{y=0}^{10}\int_{x=0}^{2y} ky(x+y)\,dx\,dy = \int_{y=0}^{10}\left(ky\left(\frac{1}{2}x^2+yx\right)\right)\Bigg|_{x=0}^{2y} dy$$

$$= \int_{y=0}^{10} ky\left(\frac{1}{2}\times 4y^2+2y^2\right) dy$$

$$= \int_{y=0}^{10} 4ky^3\,dy$$

$$= \left(ky^4\right)\Big|_0^{10}$$

But this is the total probability so it must equal 1:

$$1=\left(ky^4\right)\Big|_0^{10}=10^4k \quad\Rightarrow\quad k=\frac{1}{10^4}$$

The required probability is:

$$\int_{y=0}^{10}\int_{x=0}^{y} \frac{y(x+y)}{10^4}\,dx\,dy = \int_{y=0}^{10}\left(\frac{y}{10^4}\left(\frac{1}{2}x^2+yx\right)\right)\Bigg|_{x=0}^{y} dy$$

$$= \int_{y=0}^{10} \frac{y}{10^4}\left(\frac{1}{2}\times y^2+y^2\right) dy$$

$$= \int_{y=0}^{10} \frac{3y^3}{2\times 10^4}\,dy$$

$$= \frac{1}{2\times 10^4}\left(\frac{3}{4}y^4\right)\Big|_0^{10}$$

$$= \frac{1}{2\times 10^4}\left(\frac{3}{4}\times 10^4\right)=0.375$$

## *Solution 48*

Answer: **E**

⇨ *This topic is covered in Chapter 8.*

The formula for covariance is:

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

We need to calculate $E[X]$ and $E[Y]$:

$$\begin{aligned}
E[X] = \int_{y=0}^{10}\int_{x=0}^{2y} \frac{xy(x+y)}{10^4}\,dx\,dy &= \frac{1}{10^4}\int_{y=0}^{10}\int_{x=0}^{2y} x^2y + xy^2\,dx\,dy \\
&= \frac{1}{10^4}\int_{y=0}^{10}\left(\frac{1}{3}x^3y + \frac{1}{2}x^2y^2\right)\Bigg|_{x=0}^{2y} dy \\
&= \frac{1}{10^4}\int_{y=0}^{10} \frac{8}{3}y^4 + 2y^4\,dy \\
&= \frac{1}{10^4}\int_{y=0}^{10} \frac{14}{3}y^4\,dy \\
&= \frac{1}{10^4}\left(\frac{14}{15}y^5\right)\Bigg|_0^{10} \\
&= \frac{10^5 \times 14}{10^4 \times 15} = 9.33
\end{aligned}$$

$$\begin{aligned}
E[Y] = \int_{y=0}^{10}\int_{x=0}^{2y} \frac{y^2(x+y)}{10^4}\,dx\,dy &= \frac{1}{10^4}\int_{y=0}^{10} y^2\left(\frac{1}{2}x^2 + yx\right)\Bigg|_{x=0}^{2y} dy \\
&= \frac{1}{10^4}\int_{y=0}^{10} y^2(2y^2 + 2y^2)\,dy \\
&= \frac{1}{10^4}\int_{y=0}^{10} 4y^4\,dy \\
&= \frac{1}{10^4}\left(\frac{4}{5}y^5\right)\Bigg|_0^{10} \\
&= \frac{10^5 \times 4}{10^4 \times 5} = 8
\end{aligned}$$

Then we need $E[XY]$:

$$E[XY] = \int_{y=0}^{10}\int_{x=0}^{2y} \frac{xy^2(x+y)}{10^4}\,dx\,dy = \frac{1}{10^4}\int_{y=0}^{10}\left(\frac{1}{3}y^2x^3 + \frac{1}{2}y^3x^2\right)\Bigg|_{x=0}^{2y} dy$$

$$= \frac{1}{10^4}\int_{y=0}^{10}\left(\frac{8}{3}y^5 + \frac{4}{2}y^5\right)dy$$

$$= \frac{1}{10^4}\int_{y=0}^{10}\frac{14}{3}y^5\,dy$$

$$= \frac{1}{10^4}\left(\frac{14}{18}y^6\right)\Bigg|_0^{10}$$

$$= \frac{10^6\times 14}{10^4\times 18} = 77.78$$

Finally:

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y] = 77.78 - 8\times 9.33 = 3.1$$