Probability, Fourth Edition

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Published by BPP Professional Education

Solutions to practice questions – Chapter 9

Solution 9.1

Let X be the number of demands made each day, and let Y be the number of demands handled.

Then we have:

$$Y = \min\{X, 3\}$$

and:

$$\begin{aligned}
\Pr(Y=0) &= \Pr(X=0) = e^{-2} = 0.13534 \\
\Pr(Y=1) &= \Pr(X=1) = 2e^{-2} = 0.27067 \\
\Pr(Y=2) &= \Pr(X=2) = \frac{2^2 e^{-2}}{2!} = 0.27067 \\
\Pr(Y=3) &= \Pr(X \geq 3) = 1 - 0.13534 - 0.27067 - 0.27067 = 0.32332
\end{aligned}$$

Hence, the expected number of demands handled is:

$$\begin{aligned}
E[Y] &= 0 \times 0.13534 \ + \ 1 \times 0.27067 \ + \ 2 \times 0.27067 \ + \ 3 \times 0.32332 \\
&= 1.782
\end{aligned}$$

Solution 9.2

We can calculate $E[Y]$ and $E[Y^2]$ in terms of $E[X]$ and $E[X^2]$:

$$E[Y] = \sum_{y=0}^{\infty} y \Pr(Y=y) = \sum_{y=1}^{\infty} y \Pr(Y=y) = (1-\alpha)\sum_{y=1}^{\infty} y \Pr(X=y) = (1-\alpha)E[X] = (1-\alpha)\mu$$

$$E[Y^2] = \sum_{y=0}^{\infty} y^2 \Pr(Y=y) = \sum_{y=1}^{\infty} y^2 \Pr(Y=y) = (1-\alpha)\sum_{y=1}^{\infty} y^2 \Pr(X=y) = (1-\alpha)E[X^2] = (1-\alpha)(\mu^2+\mu)$$

Hence, the variance of Y is:

$$\begin{aligned}
\text{var}(Y) &= E[Y^2] - (E[Y])^2 \\
&= (1-\alpha)(\mu^2+\mu) - \left((1-\alpha)\mu\right)^2 \\
&= (1-\alpha)\mu(1+\alpha\mu)
\end{aligned}$$

Solution 9.3

Note that since the pdf is defined for $x > 0$, the transformation $Y = X^2$ is 1-1, with:

$$X = +\sqrt{Y}$$

So, using the method of transformations:

$$f_Y(y) = f_X\left(\sqrt{y}\right)\left|\frac{d(\sqrt{y})}{dy}\right| = e^{-\sqrt{y}} \times \frac{1}{2\sqrt{y}} \qquad \text{for } y > 0$$

Solution 9.4

The pdf of X is:

$$f_X(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\theta^{\alpha}\Gamma(\alpha)} \qquad \text{for } x > 0$$

Now, if $Y = g(X) = cX$, we have a 1-1 differentiable transformation.

The inverse transformation is:

$$X = Y/c$$

So, using the method of transformations:

$$f_Y(y) = f_X\left(\frac{y}{c}\right)\left|\frac{d(y/c)}{dy}\right| = \frac{(y/c)^{\alpha-1}e^{-(y/c)/\theta}}{\theta^{\alpha}\Gamma(\alpha)} \times \frac{1}{c}$$

$$= \frac{y^{\alpha-1}e^{-y/(c\theta)}}{(c\theta)^{\alpha}\Gamma(\alpha)} \qquad \text{for } y > 0$$

Hence Y follows a gamma distribution with parameters α and $c\theta$.

Solution 9.5

Let's use the method of distribution functions.

We have:

$$F_Y(y) = \Pr(Y \le y) = \Pr\left(\frac{1}{X} - 1 \le y\right) = \Pr\left(X \ge \frac{1}{1+y}\right) = 1 - \Pr\left(X < \frac{1}{1+y}\right) = 1 - \frac{1}{1+y}$$

Hence, the pdf of Y is:

$$f_Y(y) = F_Y'(y) = \frac{1}{(1+y)^2} \qquad \text{for } y > 0$$

Solution 9.6

The density of T is:

$$f(t) = \frac{1}{3}e^{-t/3} \qquad \text{for } t > 0$$

The discovery time is a piecewise function of T:

$$X = g(T) = \max(T, 2) = \begin{cases} 2 & T \le 2 \\ T & T > 2 \end{cases}$$

Hence:

$$E[X] = \int_0^\infty g(t) f(t)\, dt = 2\int_0^2 \frac{e^{-t/3}}{3} dt + \int_2^\infty \frac{t\, e^{-t/3}}{3} dt$$

$$= 2 \times F(2) + \left(-t\, e^{-t/3} - 3\, e^{-t/3}\right)\Big|_2^\infty \qquad \text{(integrating by parts)}$$

$$= 2\left(1 - e^{-2/3}\right) + \left(-0 - 0 + 2\, e^{-2/3} + 3\, e^{-2/3}\right)$$

$$= 2 + 3\, e^{-2/3}$$

Solution 9.7

We are given that T is distributed uniformly on the interval $[8,12]$, so the probability density function is:

$$f_T(t) = \frac{1}{4} \qquad \text{for } 8 \le t \le 12$$

The random rate of service in customers per minute is:

$$R = 10/T$$

Since this is a 1-1, differentiable transformation of T with inverse function $T = 10/R$, we can calculate the probability density function $f_R(r)$ using the method of transformations:

$$f_R(r) = f_T\left(g^{-1}(r)\right)\left|\frac{dg^{-1}(r)}{dr}\right|$$

$$= f_T\left(\frac{10}{r}\right) \times \left|\frac{d(10/r)}{dr}\right|$$

$$= \frac{1}{4} \times \left|\frac{-10}{r^2}\right|$$

$$= \frac{5}{2r^2}$$

Solution 9.8

Let Y be the length of time lived in the next 10 years.

Then Y is defined as follows:

$$Y = \min(X,10) = \begin{cases} X & 0 \le X \le 10 \\ 10 & X > 10 \end{cases}$$

The expected value of Y is therefore:

$$E[Y] = \int g(x) f(x) \, dx$$

$$= \int_0^{10} x \frac{1}{75} dx + \int_{10}^{75} 10 \frac{1}{75} dx$$

$$= \left. \frac{x^2}{150} \right|_0^{10} + \left. \frac{10x}{75} \right|_{10}^{75}$$

$$= 9.333$$

Note: This question uses the theory related to policy limits, since we apply a maximum of 10 to the future lifetime of the 25-year-old.

Solution 9.9

The benefit amount paid by the insurer is:

$$Y = \max\{0,(X-250)\} = \begin{cases} 0 & 0 \le X \le 250 \\ X-250 & X > 250 \end{cases}$$

So the expected benefit amount paid by the insurer is:

$$E[Y] = \int_0^{250} 0 \times f_X(x) \, dx + \int_{250}^{\infty} (x-250) f_X(x) dx$$

$$= 0 + \int_{250}^{\infty} (x-250) \frac{1}{1{,}000} e^{-x/1{,}000} dx$$

We could use integration by parts to finish the calculation. However, it is quite a bit quicker to employ the substitution $z = x - 250$:

$$E[Y] = \int_0^{\infty} z \frac{1}{1{,}000} e^{-(z+250)/1{,}000} \, dz$$

$$= e^{-250/1{,}000} \int_0^{\infty} z \frac{1}{1{,}000} e^{-z/1{,}000} dz$$

Finally, note that the last integral is the expected value of an exponential distribution with mean 1,000. So without performing any further integration, we can see that the final result is:

$$E[Y] = e^{-0.25} \times 1{,}000 = 778.80$$

Solution 9.10

Let X be the manufacturer's annual losses, and let Y be the part of the annual losses not paid by the insurance company. Then:

$$Y = \min(X, 2) = \begin{cases} X & 0.6 < X \le 2 \\ 2 & X > 2 \end{cases}$$

We can calculate the expected value of Y as:

$$E[Y] = \int_{0.6}^{2} x\, f(x)\, dx + \int_{2}^{\infty} 2\, f(x)\, dx$$

$$= 2.5(0.6)^{2.5} \int_{0.6}^{2} x^{-2.5}\, dx + 2 \times 2.5(0.6)^{2.5} \int_{2}^{\infty} x^{-3.5}\, dx$$

$$= -\frac{2.5(0.6)^{2.5}}{1.5 x^{1.5}}\Bigg|_{0.6}^{2} - \frac{2.(0.6)^{2.5}}{x^{2.5}}\Bigg|_{2}^{\infty}$$

$$= \frac{2.5}{1.5}\left(0.6 - \frac{(0.6)^{2.5}}{(2)^{1.5}}\right) + 2\left(\frac{0.6}{2}\right)^{2.5}$$

$$= 0.9343$$

Solution 9.11

Since $Y = 1.05X$ is a 1-1 differentiable transformation, we can use the method of transformations.

First, we identify the inverse function of the transformation. This is simply:

$$x = y / 1.05$$

Hence, the density function of Y, the loss in 2003, is given by:

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right| = f_X(y/1.05)\left|\frac{d(y/1.05)}{dy}\right|$$

$$= \frac{2 \times 1{,}000^2}{\left((y/1.05) + 1{,}000\right)^3} \times \frac{1}{1.05}$$

$$= \frac{2 \times 1{,}050^2}{(y + 1{,}050)^3} \qquad \text{for } y > 0$$

Solution 9.12

Let $Y = X_1 + X_2$ where X_1 and X_2 are the lifetimes of the primary and backup systems.

The joint density of X_1 and X_2 is:

$$f(x_1, x_2) = f(x_1) f(x_2) = 0.05\, e^{-0.05x_1} \times 0.10\, e^{-0.10x_2} \qquad \text{where } x_1, x_2 > 0$$

Step 1: The region in the $X_1 \times X_2$ plane defined by the inequality $X_1 + X_2 \le y$ is a triangular region in the first quadrant below the line $X_1 + X_2 = y$.

Step 2: The cdf of Y is calculated as:

$$F_Y(y) = \Pr(X_1 + X_2 \le y) = \int_{x_2=0}^{y} \int_{x_1=0}^{y-x_2} 0.05\, e^{-0.05x_1}\, 0.10\, e^{-0.10x_2}\, dx_1\, dx_2$$

$$= \int_{x_2=0}^{y} 0.10\, e^{-0.10x_2} \left(1 - e^{-0.05(y-x_2)}\right) dx_2$$

$$= \int_0^y 0.10 e^{-0.10x_2}\, dx_2 - 2e^{-0.05y} \int_0^y 0.05 e^{-0.05x_2}\, dx_2$$

$$= \left(1 - e^{-0.10y}\right) - 2e^{-0.05y}\left(1 - e^{-0.05y}\right)$$

$$= 1 + e^{-0.10y} - 2e^{-0.05y}$$

Step 3: Differentiate this to obtain the density function of Y:

$$f_Y(y) = F_Y'(y) = 0.1\left(e^{-0.05y} - e^{-0.10y}\right)$$

Solution 9.13

We need:

$$F_X(x) = \Pr(X \le x) = 1 - \left(\frac{50}{x+50}\right)^2$$

and:

$$f_X(x) = \frac{d}{dx}\left[1 - \left(\frac{50}{x+50}\right)^2\right] = -\frac{d}{dx} 50^2 (x+50)^{-2} = \frac{5,000}{(x+50)^3}$$

Then the pdf of the 3rd largest claim (which is also the 23rd smallest claim) is:

$$f_{Y_{23}}(y) = \frac{25!}{22!2!} \left(F_X(y)\right)^{22} \left(1 - F_X(y)\right)^2 f_X(y)$$

$$= 6,900 \times \left[1 - \left(\frac{50}{y+50}\right)^2\right]^{22} \times \left[\left(\frac{50}{y+50}\right)^2\right]^2 \times \frac{5,000}{(y+50)^3}$$

$$= \frac{2.15625 \times 10^{14}}{(y+50)^7} \times \left[1 - \left(\frac{50}{y+50}\right)^2\right]^{22}$$

Solution 9.14

For an exponential distribution with mean 10, we have:

$$f(x) = 0.1e^{-0.1x}$$
$$F(x) = 1 - e^{-0.1x}$$

Hence, the pdf of the 4th largest claim (which is also the 2nd smallest claim) is:

$$f_{Y_2}(y) = \frac{5!}{1!3!}\left(F_X(y)\right)^1 \left(1 - F_X(y)\right)^3 f_X(y)$$
$$= 20 \times \left(1 - e^{-0.1y}\right) \times \left(e^{-0.1y}\right)^3 \times 0.1e^{-0.1y}$$
$$= 2\left(e^{-0.4y} - e^{-0.5y}\right)$$

Hence the expected value is:

$$\int_0^\infty y f_{Y_2}(y)dy = \int_0^\infty 2y\left(e^{-0.4y} - e^{-0.5y}\right)dy$$
$$= 5\int_0^\infty 0.4ye^{-0.4y}dy - 4\int_0^\infty 0.5ye^{-0.5y}dy$$

Notice that these two integrals are the expected values of exponential distributions with means 2.5 and 2 , hence:

$$E[Y_2] = 5 \times 2.5 - 4 \times 2 = 4.5$$

Solution 9.15

The first step here is to calculate the density function of:

$$Y = \max\{X_1, X_2, X_3\}$$

We can use the method of distribution functions to calculate the distribution function of Y as follows:

$$F_Y(y) = \Pr(Y \le y) = \Pr(all\ X_i \le y) = \left(F_X(y)\right)^3$$
$$= \left(\int_1^y \frac{3}{x^4}dx\right)^3 = \left(1 - \frac{1}{y^3}\right)^3 \qquad y > 1$$

We differentiate the cdf to find the pdf:

$$f_Y(y) = F_Y'(y) = 3\left(1 - \frac{1}{y^3}\right)^2 \frac{3}{y^4} = 9\left(\frac{1}{y^4} - \frac{2}{y^7} + \frac{1}{y^{10}}\right) \qquad y > 1$$

and then compute $E[Y]$ as:

$$E[Y] = \int_1^\infty y f_Y(y)dy = 9\int_1^\infty \frac{1}{y^3} - \frac{2}{y^6} + \frac{1}{y^9}dy$$
$$= 9\left(-\frac{1}{2y^2} + \frac{2}{5y^5} - \frac{1}{8y^8}\right)\Bigg|_1^\infty = 9\left(\frac{1}{2} - \frac{2}{5} + \frac{1}{8}\right) = 2.025 \text{ (thousand)}$$