Probability, Fourth Edition

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Solutions to practice questions – Chapter 7

Solution 7.1

(i) $\Pr(X > 3.1) = 1 - \Phi(3.1) = 1 - 0.9990 = 0.0010$

(ii) $\Pr(X < -1.4) = \Phi(-1.4) = 1 - \Phi(1.4) = 1 - 0.9192 = 0.0808$

(iii) $\Pr(0.4 < X < 2.2) = \Phi(2.2) - \Phi(0.4) = 0.9861 - 0.6554 = 0.3307$

(iv) $\Pr(-1.7 < X < -0.2) = \Phi(-0.2) - \Phi(-1.7) = (1 - \Phi(0.2)) - (1 - \Phi(1.7))$

$$= \Phi(1.7) - \Phi(0.2) = 0.9554 - 0.5793 = 0.3761$$

Solution 7.2

The moment generating function of X is:

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Hence the cumulant generating function of X is:

$$R_X(t) = \ln\left(M_X(t)\right) = \mu t + \frac{\sigma^2 t^2}{2}$$

(i) The expected value of X is:

$$E[X] = R'_X(0) = \mu + 0 \times \sigma^2 = \mu$$

(ii) The variance of X is:

$$\operatorname{var}(X) = R''_X(0) = \sigma^2$$

Solution 7.3

The moment generating function is:

$$\exp\left(8(t^2 - 1.5t)\right) = \exp\left(-12t + \frac{16t^2}{2}\right)$$

Comparing this to the MGF of a normal distribution:

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

We can see that $X \sim N(-12, 16)$.

Solution 7.4

By Theorem 7.1, $\left(\frac{X-15}{\sqrt{100}}\right) \sim N(0,1)$. Hence:

(i) $\Pr(X > 18) = 1 - \Pr(X \le 18) = 1 - \Pr\left(\frac{X-15}{10} \le \frac{18-15}{10}\right) = 1 - \Phi(0.3) = 1 - 0.6179 = 0.3821$

(ii) $\Pr(3 < X < 27) = \Pr\left(\frac{3-15}{10} < \frac{X-15}{10} < \frac{27-15}{10}\right) = \Phi(1.2) - \Phi(-1.2)$

$$= \Phi(1.2) - \left(1 - \Phi(1.2)\right) = 2 \times \Phi(1.2) - 1 = 2 \times 0.8849 - 1 = 0.7698$$

(iii) $\Pr(-4 < X < 4) = \Pr\left(\frac{-4-15}{10} < \frac{X-15}{10} < \frac{4-15}{10}\right) = \Phi(-1.1) - \Phi(-1.9)$

$$= \left(1 - \Phi(1.1)\right) - \left(1 - \Phi(1.9)\right) = \Phi(1.9) - \Phi(1.1) = 0.9713 - 0.8643 = 0.1070$$

Solution 7.5

The probability that a sample value (X) is less than 9 is:

$$\Pr(X < 9) = \Pr\left(\frac{X-10}{\sqrt{2}} < \frac{9-10}{\sqrt{2}}\right) = \Phi(-0.7071) = 1 - \Phi(0.7071) = 1 - 0.7601 = 0.2399$$

This last value is found using linear interpolation:

$$\Phi(0.7071) \approx \Phi(0.7) + \frac{0.7071 - 0.7}{0.8 - 0.7} \times \left[\Phi(0.8) - \Phi(0.7)\right] = 0.7601$$

Hence, the required probability is:

$$\left({}_6C_2\right)(0.2399)^2 (0.7601)^4 = 0.2882$$

Solution 7.6

Let X_1 and X_2 be the sizes of the two claims, and let $Y = X_1 - X_2$ be the difference between these random variables. Then by Theorem 7.4 (with $a_1 = 1$ and $a_2 = -1$), we have:

$$Y \sim N\left((1{,}800 - 1{,}800)\,,\,(400^2 + 400^2)\right) = N(0\ ,\ 320{,}000)$$

The probability that the claims differ by more than \$500 is:

$$\Pr(Y < -500) + \Pr(Y > 500) = 2 \times \Pr(Y > 500) \quad \text{by symmetry}$$

We have:

$$\Pr(Y > 500) = 1 - \Pr(Y \le 500) = 1 - \Pr\left(\frac{Y-0}{\sqrt{320{,}000}} \le \frac{500-0}{\sqrt{320{,}000}}\right)$$

$$= 1 - \Phi(0.8839) = 1 - 0.8114 = 0.1886$$

Hence the required probability is:

$$\Pr(Y < -500) + \Pr(Y > 500) = 2 \times \Pr(Y > 500) = 2 \times 0.1886 = 0.3772$$

Solution 7.7

Let X_i be the lifetime of the i-th bulb. A succession of n bulbs will produce light for a random time equal to:

$$S = X_1 + \cdots + X_n$$

Since $X_i \sim N(3,1)$ and the lifetimes are independent, we know that S is normally distributed and:

$$E[S] = n\,E[X] = 3n$$

$$\text{var}(S) = n\,\text{var}(X) = n$$

We want to find the smallest n such that $\Pr(S > 40) \ge 0.9772$.

The first step is to convert this probability statement to one about the standard normal distribution:

$$\Pr(S > 40) = \Pr\left(\frac{S-3n}{\sqrt{n}} > \frac{40-3n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{40-3n}{\sqrt{n}}\right)$$

From the normal distribution table, note that:

$$\Phi(2) = 0.9772$$
$$\Rightarrow 1 - \Phi(-2) = 0.9772 \quad \text{(by symmetry)}$$

Hence:

$$\frac{40-3n}{\sqrt{n}} = -2 \quad \Rightarrow 3n - 2\sqrt{n} - 40 = 0$$

We can view this equation as a quadratic in the variable $\sqrt{n}$.

By the quadratic formula:

$$\sqrt{n} = \frac{-(-2) \pm \sqrt{4 - 4 \times 3 \times (-40)}}{2 \times 3} = \frac{2 \pm 22}{6}$$

$$\Rightarrow \sqrt{n} = \frac{24}{6} = 4 \quad \text{(positive root)}$$

$$\Rightarrow n = 16$$

Note: If, for example, the solution of the quadratic equation were $n = 16.13$, then the answer would be $n = 17$ and the probability would be slightly higher than 0.9772.

Solution 7.8

Let the less accurate measurement be X and let the more accurate measurement be Y. Then:

$$X \sim N\left(h, (0.0056h)^2\right)$$

$$Y \sim N\left(h, (0.0044h)^2\right)$$

Let Z be the average value of X and Y. Then Z is normally distributed with:

$$\mu = E[Z] = E\left[\frac{X+Y}{2}\right] = \frac{1}{2}\left(E[X] + E[Y]\right) = \frac{1}{2}(h+h) = h$$

$$\sigma^2 = \text{var}(Z) = \text{var}\left(\frac{X+Y}{2}\right) = \left(\frac{1}{2}\right)^2 \text{var}(X+Y) = \frac{1}{4}\left(\text{var}(X) + \text{var}(Y)\right) = 0.00001268h^2$$

$$\Rightarrow \sigma = 0.003561h$$

To calculate the probability that the average value is within $0.005h$ of the actual height h:

$$\Pr(0.995h < Z < 1.005h) = \Pr\left(\frac{0.995h - h}{0.003561h} < \frac{Z-h}{0.003561h} < \frac{1.005h - h}{0.003561h}\right)$$

$$= \Pr\left(-1.4041 < \frac{Z-h}{0.00356h} < 1.4041\right)$$

$$= \Phi(1.4041) - \Phi(-1.4041)$$

$$= \Phi(1.4041) - \left(1 - \Phi(1.4041)\right)$$

$$= 2\Phi(1.4041) - 1 = 2 \times 0.9198 - 1 = 0.8396$$

Solution 7.9

Let X_i be the amount of the i-th claim. The total claim amount for the 200 claims is:

$$S = X_1 + X_2 + \cdots + X_{200}$$

Since $X_i \sim N\left(237, 202^2\right)$ and the claims are independent, then S is normally distributed with:

$$E[S] = 200 \times 237 = 47,400$$

$$\text{var}(S) = 200 \times 202^2 = 8,160,800$$

The required probability is:

$$\Pr(S > 50,000) = 1 - \Pr(S \le 50,000) = 1 - \Pr\left(\frac{S - 47,400}{\sqrt{8,160,800}} \le \frac{50,000 - 47,400}{\sqrt{8,160,800}}\right)$$

$$= 1 - \Phi(0.9101) = 1 - 0.8185 = 0.1815$$

Solution 7.10

Let X_i be the amount of the i-th claim. The average amount of the 80 claims is:

$$\bar{X} = \frac{1}{80}(X_1 + X_2 + \cdots + X_{80})$$

Since $X_i \sim N\left(574, 186^2\right)$ and the claims are independent, then $\bar{X}$ is normally distributed with:

$$E\left[\bar{X}\right] = 574 \qquad \text{var}\left(\bar{X}\right) = \frac{1}{80} \times 186^2 = 432.45$$

The required probability is:

$$\Pr\left(\bar{X} < 565\right) = \Pr\left(\frac{\bar{X} - 574}{\sqrt{432.45}} \le \frac{565 - 574}{\sqrt{432.45}}\right) = \Phi(-0.4328) = 1 - \Phi(0.4328) = 1 - 0.6672 = 0.3328$$

Solution 7.11

You should recognize that the given density is exponential with $\theta = 1,000$. Therefore, the mean and variance of the claim amount per policy, X, are:

$$E[X] = \theta = 1,000 \qquad \text{var}(X) = \theta^2 = 1,000,000$$

The total claims for the group of 100 policies can be written as:

$$S = X_1 + \cdots + X_{100}$$

where the various X_i are identically distributed like X.

The phrase in the question "the *approximate* probability" is a clue to use the Central Limit Theorem, even though the question does not state that the claim amounts for different policies are independent. If we assume independence, then:

$$E[S] = 100\,E[X] = 100,000$$

$$\text{var}(S) = 100\,\text{var}(X) = 100,000,000 \quad \Rightarrow \quad \sigma_S = \sqrt{\text{var}(S)} = 10,000$$

By the Central Limit Theorem, S is approximately normal in distribution and:

$$\frac{S-E[S]}{\sigma_S}=\frac{S-100,000}{10,000}\sim N(0,1)$$

The premium for a single policy is:

$$100+E[X]=100+1,000=1,100$$

The premium for the group of 100 policies is thus 110,000. So, the probability that claims exceed premium is given by:

$$\Pr(S>110,000)=\Pr\left(\frac{S-100,000}{10,000}>\frac{110,000-100,000}{10,000}\right)$$

$$=\Pr\left(\frac{S-100,000}{10,000}>1\right)\approx 1-\Phi(1)=1-0.8413=0.1587$$

Solution 7.12

We can make use of results in Solution 7.11.

Let G be the group premium. The value of G is set such that:

$$\Pr(S>G)=0.1$$

$$\Rightarrow\quad \Pr(S\le G)=0.9$$

$$\Rightarrow\quad \Pr\left(\frac{S-100,000}{10,000}\le\frac{G-100,000}{10,000}\right)=0.9$$

$$\Rightarrow\quad \Phi\left(\frac{G-100,000}{10,000}\right)=0.9$$

$$\Rightarrow\quad \frac{G-100,000}{10,000}=1.282$$

Solving for G allows us to calculate the premium per policy:

$$\frac{G-100,000}{10,000}=1.282\quad\Rightarrow\quad G=112,820$$

$$\Rightarrow\quad \text{Premium per policy}=\frac{G}{100}=1,128.20$$

Solution 7.13

Let X_i be the difference between the true age and the rounded age of the i-th individual.

Since X_i is distributed uniformly on the interval $[-2.5,2.5]$, it follows that:

$$E[X_i]=\frac{-2.5+2.5}{2}=0$$

$$\operatorname{var}(X_i)=\frac{(2.5-(-2.5))^2}{12}=\frac{25}{12}=2.08333$$

Assuming that $X_1, X_2, \cdots, X_{48}$ are independent, then using the Central Limit Theorem:

$$\bar{X} = \frac{1}{48}\sum_{i=1}^{48} X_i \text{ is approximately distributed } N\left(0, \frac{1}{48} \times 2.08333\right) = N(0, 0.0434)$$

Finally, we have:

$$\Pr\left(-0.25 < \bar{X} < 0.25\right) = \Pr\left(\frac{-0.25-0}{\sqrt{0.0434}} < \frac{\bar{X}-0}{\sqrt{0.0434}} < \frac{0.25-0}{\sqrt{0.0434}}\right) = \Pr\left(-1.2 < \frac{\bar{X}-0}{\sqrt{0.0434}} < 1.2\right)$$

$$\approx \Phi(1.20) - \Phi(-1.20) = \Phi(1.20) - \left(1 - \Phi(1.20)\right)$$

$$= 2\,\Phi(1.20) - 1 = 2 \times 0.8849 - 1 = 0.7698$$

Solution 7.14

The number of sales, N, follows a binomial distribution with parameters $n = 200$ and $p = 0.7$. Hence:

$$E[N] = np = 140 \qquad \text{var}(N) = npq = 42$$

We can approximate the distribution of N with the random variable $X \sim N(140, 42)$.

Hence the required probability is:

$$\Pr(124 < N \le 153) \approx \Pr(124.5 < X < 153.5) = \Pr\left(\frac{124.5-140}{\sqrt{42}} < \frac{X-140}{\sqrt{42}} < \frac{153.5-140}{\sqrt{42}}\right)$$

$$= \Phi(2.0831) - \Phi(-2.3917) = \Phi(2.0831) - \left(1 - \Phi(2.3917)\right)$$

$$= 0.9813 - (1 - 0.9916) = 0.9729$$

Solution 7.15

By the additive property of the Poisson distribution, the total number of claims, N, follows a Poisson distribution with mean $0.3 \times 1{,}000 = 300$. Hence:

$$E[N] = 300 \qquad \text{var}(N) = 300$$

We can approximate the distribution of N with the random variable $X \sim N(300, 300)$.

Hence the required probability is:

$$\Pr(N \ge 340) \approx \Pr(X > 339.5) = 1 - \Pr(X \le 339.5) = 1 - \Pr\left(\frac{X-300}{\sqrt{300}} \le \frac{339.5-300}{\sqrt{300}}\right)$$

$$= 1 - \Phi(2.2805) = 1 - 0.9887 = 0.0113$$

Solution 7.16

We need:

$$E[Y] = E\left[e^X\right]$$

which is the value of the moment generating function of X when $t = 1$.

So:

$$E[Y] = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Bigg|_{t=1} = \exp\left(\mu + \frac{1}{2}\sigma^2\right) = \exp\left(0.5 + \frac{1}{2} \times 0.16\right) = \exp(0.58) = 1.7860$$

Solution 7.17

The variance of Y is:

$$\text{var}[Y] = E\left[Y^2\right] - (E[Y])^2 = E\left[e^{2X}\right] - \left(E\left[e^X\right]\right)^2 = M_X(2) - (M_X(1))^2$$

$$= \exp\left(2\mu + 2\sigma^2\right) - \exp\left(2\mu + \sigma^2\right)$$

because, for the normal distribution, $M_X(t) = \exp\left(\mu t + \dfrac{\sigma^2 t^2}{2}\right)$.

Hence:

$$\text{var}[Y] = \exp\left(2 \times 2.5 + 2 \times 1.6^2\right) - \exp\left(2 \times 2.5 + 1.6^2\right) = e^{10.12} - e^{7.56} = 22{,}915$$

Solution 7.18

We need:

$$E[Y] = 10{,}000\, E\left[e^X\right]$$

where $E\left[e^X\right]$ is the value of the moment generating function of X when $t = 1$.

So:

$$E[Y] = 10{,}000 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Bigg|_{t=1} = 10{,}000 \exp\left(\mu + \frac{1}{2}\sigma^2\right) = 10{,}000 \exp\left(0.06 + \frac{0.015^2}{2}\right) = e^{0.0601125}$$

$$= 10{,}619.56$$

Solution 7.19

We need:

$$\begin{aligned}\Pr\left(1{,}000e^{\Delta} < 500\right) &= \Pr\left(e^{\Delta} < 0.5\right)\\ &= \Pr(\Delta < \ln 0.5)\\ &= \Pr\left(Z < \frac{\ln 0.5 - 0.1}{\sqrt{0.16}}\right)\\ &= \Pr(Z < -1.9829) = \Pr(Z > 1.9829) = 1 - \Phi(1.9829) = 0.0238\end{aligned}$$

by interpolating between the tabular values of $\Phi(1.9)$ and $\Phi(2.0)$ in Table 7.1.

Solution 7.20

First we need:

$$E[Y] = E\left[e^{0.5X}\right] = M_X(0.5)$$

For the normal distribution:

$$M_X(0.5) = \exp\left(0.5\mu + \frac{(0.5)^2\sigma^2}{2}\right) = \exp(0.5\times 10 + 0.125\times 16) = e^7$$

So now we need:

$$\Pr\left(Y > e^7\right) = \Pr\left(e^{0.5X} > e^7\right) = \Pr(0.5X > 7) = \Pr(X > 14)$$

$$= \Pr\left(Z > \frac{14-10}{4}\right) = \Pr(Z > 1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$