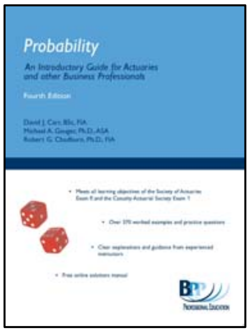

Probability, Fourth Edition

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Solutions to practice questions – Chapter 6

Solution 6.1

Let the losses due to storm, fire, and theft be denoted X_1, X_2 and X_3 respectively.

$$\begin{aligned}\Pr\left(\max\left\{X_1, X_2, X_3\right\} > 2.5\right) &= 1 - \Pr\left(\max\left\{X_1, X_2, X_3\right\} \le 2.5\right) \\ &= 1 - \Pr\left(X_1 \le 2.5\right) \times \Pr\left(X_2 \le 2.5\right) \times \Pr\left(X_3 \le 2.5\right)\end{aligned}$$

The distribution function for a uniform distribution with range $[a, b]$ is:

$$F(x) = \frac{x-a}{b-a}$$

So, we can calculate the probability as:

$$\begin{aligned}\Pr\left(\max\left\{X_1, X_2, X_3\right\} > 2.5\right) &= 1 - \Pr\left(X_1 \le 2.5\right) \times \Pr\left(X_2 \le 2.5\right) \times \Pr\left(X_3 \le 2.5\right) \\ &= 1 - (1)\left(\frac{2.5}{3}\right)\left(\frac{2.5}{4}\right) = 1 - 0.521 = 0.479\end{aligned}$$

Solution 6.2

We have:

$$E[X] = \frac{a+b}{2} = 1{,}879$$

$$\operatorname{var}(X) = \frac{(b-a)^2}{12} = 507 \quad \Rightarrow \quad b - a = \sqrt{12 \times 507} = 78$$

Hence:

$$\begin{aligned}&a + b = 2 \times 1{,}879 = 3{,}758 \\ &\quad \Rightarrow \quad a + (78 + a) = 3{,}758 \\ &\quad \Rightarrow \quad a = 1{,}840 \\ &\quad \Rightarrow \quad b = 1{,}918\end{aligned}$$

Solution 6.3

We need $x_{0.9}$ such that:

$$0.9 = \int_0^{x_{0.9}} f(x)\,dx = 375\int_0^{x_{0.9}} (x+5)^{-4}\,dx = -\frac{375}{3}(x+5)^{-3}\Big|_0^{x_{0.9}} = 125\left(\frac{1}{125} - \frac{1}{(x_{0.9}+5)^3}\right)$$

$$\Rightarrow \frac{125}{(x_{0.9}+5)^3} = 0.1 \Rightarrow x_{0.9} = (1{,}250)^{1/3} - 5 = 5.772$$

Solution 6.4

We need:

$$E[X] = \int_0^{\infty} x\,3\times 2{,}000^3\,(x+2{,}000)^{-4}\,dx$$

Integrating by parts:

$$\begin{aligned} E[X] &= -\frac{x\,2{,}000^3}{(x+2{,}000)^3}\Big|_0^{\infty} + \int_0^{\infty} 2{,}000^3\,(x+2{,}000)^{-3}\,dx \\ &= 0 - \frac{2{,}000^3}{2\,(x+2{,}000)^2}\Big|_0^{\infty} \\ &= 1{,}000 \end{aligned}$$

Solution 6.5

We want to calculate the following:

$$\Pr\left(X<2 \middle| X>1.5\right) = \frac{\Pr(X<2 \cap X>1.5)}{\Pr(X>1.5)} = \frac{\Pr(1.5<X<2)}{\Pr(X>1.5)}$$

Now:

$$\Pr(X>1.5) = \int_{1.5}^{\infty} 3x^{-4}\,dx = -\frac{1}{x^3}\Big|_{1.5}^{\infty} = \frac{1}{1.5^3} = 0.296296$$

$$\Pr(1.5<X<2) = \int_{1.5}^{2} 3x^{-4}\,dx = -\frac{1}{x^3}\Big|_{1.5}^{2} = \frac{1}{1.5^3} - \frac{1}{2^3} = 0.171296$$

So:

$$\Pr\left(X<2 \middle| X>1.5\right) = \frac{0.171296}{0.296296} = 0.578125 \quad \text{(exactly)}$$

Solution 6.6

First we need the value of c. This is found from:

$$\int_3^\infty c\,x^{-4}\,dx = 1$$

$$\Rightarrow \frac{1}{c} = -\frac{1}{3x^3}\bigg|_3^\infty = \frac{1}{81}$$

$$\Rightarrow c = 81$$

The expected lifetime is then:

$$E[X] = \int_3^\infty x\,81\,x^{-4}\,dx = -\frac{81}{2x^2}\bigg|_3^\infty = 4.5$$

Solution 6.7

For the exponential distribution with mean 50, we have:

$$\Pr(X > x) = 1 - F(x) = e^{-x/50}$$

Hence:

$$\Pr\left(X > 200 \middle| X > 50\right) = \frac{\Pr(X > 200)}{\Pr(X > 50)} = \frac{e^{-200/50}}{e^{-50/50}} = \frac{e^{-4}}{e^{-1}} = e^{-3} = 0.0498$$

Alternatively, using the memoryless property we have:

$$\Pr\left(X > 200 \middle| X > 50\right) = \Pr(X > 150) = e^{-150/50} = e^{-3} = 0.0498$$

Solution 6.8

Let X be the random time (in days) until the high-risk driver has an accident. Since X follows an exponential distribution, the cdf is:

$$F(x) = 1 - e^{-x/\theta}$$

We are given that

$$0.30 = \Pr(X < 50) = 1 - e^{-50/\theta}$$

$$\Rightarrow \quad e^{-50/\theta} = 0.70$$

Hence:

$$\Pr(X < 80) = 1 - e^{-80/\theta} = 1 - \left(e^{-50/\theta}\right)^{8/5} = 1 - 0.70^{8/5} = 0.4349$$

Note: It is unnecessary to calculate the precise value of θ. If you did solve for θ, you should find that $\theta = 140.184$.

Solution 6.9

Let T denote the random time required to repair the machine. It is assumed to be exponentially distributed with mean $\theta_T = 2$. Let X denote the cost of replacement parts. It is assumed to be gamma distributed with parameters α and θ_X such that:

$$100 = E[X] = \alpha\,\theta_X$$

$$5{,}000 = \operatorname{var}(X) = \alpha\,\theta_X^2$$

Solving these equations results in:

$$\alpha = 2 \qquad \theta_X = 50$$

We are asked to calculate the probability of the event $\{T > 3\} \cup \{X > 150\}$.

We can calculate the two components as follows:

$$\Pr(T > 3) = 1 - \Pr(T < 3) = e^{-3/2} = 0.2231$$

$$\Pr(X > 150) = 1 - \Pr(X < 150) = e^{-150/50}\left(1 + \frac{150}{50}\right) = 0.1991$$

By the additive probability law, we have:

$$\begin{aligned}\Pr(\{T>3\}\cup\{X>150\}) &= \Pr(T>3) + \Pr(X>150) - \Pr(\{T>3\}\cap\{X>150\}) \\ &= \Pr(T>3) + \Pr(X>150) - \Pr(T>3)\Pr(X>150) \\ &= 0.2231 + 0.1991 - 0.2231 \times 0.1991 = 0.3778\end{aligned}$$

Solution 6.10

We can calculate the required probability as follows:

$$\Pr(4<S<8) = \Pr(4<S<8 \cap N=0) + \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1)$$

If there are no claims (*ie* $N = 0$), then the claim amount must be zero, so:

$$\Pr(4<S<8 \cap N=0) = 0$$

Therefore:

$$\Pr(4<S<8) = \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1)$$

So from basic laws of probability, we have:

$$\begin{aligned}\Pr(4<S<8) &= \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1) \\ &= \Pr(4<S<8 \mid N=1) \times \Pr(N=1) + \Pr(4<S<8 \mid N>1) \times \Pr(N>1) \\ &= \underbrace{(F(8)-F(4))}_{\text{exp, }\theta=5} \times \tfrac{1}{3} + \underbrace{(F(8)-F(4))}_{\text{exp, }\theta=8} \times \tfrac{1}{6} \\ &= \left((1-e^{-8/5}) - (1-e^{-4/5})\right) \times \tfrac{1}{3} + \left((1-e^{-8/8}) - (1-e^{-4/8})\right) \times \tfrac{1}{6} \\ &= 0.1223\end{aligned}$$

Solution 6.11

The waiting time between accidents follows an exponential distribution with mean $\theta = \frac{5}{2} = 2.5$ days.

Hence:

$$\Pr(X > 3) = 1 - F(3) = e^{-\frac{3}{2.5}} = e^{-1.2} = 0.3012$$

Solution 6.12

From the form of the pdf, we can see that X follows a gamma distribution with $\alpha = 6$ and $\theta = 100$.
Hence:

$$E[X] = \alpha\theta = 6 \times 100 = 600$$

Solution 6.13

First, we'll determine the parameter values:

$$15 = E[X] = \alpha\,\theta$$

$$75 = \text{var}(X) = \alpha\,\theta^2$$

$$\Rightarrow \quad \theta = \frac{\text{var}(X)}{E[X]} = 5 \quad \Rightarrow \quad \alpha = 3$$

Since α is a positive integer, the cdf can be written as:

$$F(x) = 1 - e^{-x/5}\left(1 + \frac{x}{5} + \frac{x^2}{2!5^2}\right)$$

Finally:

$$\Pr(X > 30 \mid X > 15) = \frac{\Pr(X > 30)}{\Pr(X > 15)} = \frac{1 - F(30)}{1 - F(15)} = \frac{1 - 0.93803}{1 - 0.57681} = \frac{0.06197}{0.42319} = 0.1464$$

Solution 6.14

From the form of the moment generating function, we can see that X follows a gamma distribution with parameters $\alpha = 2$ and $\theta = 3$. Hence:

$$\text{var}(X) = \alpha\theta^2 = 2 \times 3^2 = 18$$

Solution 6.15

Differentiating the cumulant generating function, we have:

$$R_X(t) = \ln M_X(t) = \ln\left((1-\theta t)^{-\alpha}\right) = -\alpha \ln(1-\theta t)$$

$$R'_X(t) = \frac{\alpha\theta}{(1-\theta t)}$$

$$R''_X(t) = \frac{\alpha\theta^2}{(1-\theta t)^2} \quad \Rightarrow \quad R''_X(0) = \alpha\theta^2$$

$$R'''_X(t) = \frac{2\alpha\theta^3}{(1-\theta t)^3} \quad \Rightarrow \quad R'''_X(0) = 2\alpha\theta^3$$

Hence the skewness is:

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \frac{R'''_X(0)}{\left(R''_X(0)\right)^{3/2}} = \frac{2\alpha\theta^3}{\alpha^{3/2}\theta^3} = \frac{2}{\sqrt{\alpha}}$$

Solution 6.16

The required probability is:

$$\Pr(X > 10) = \int_{10}^{\infty} f(x)\,dx = \frac{(2)^{1.25}}{0.8}\int_{10}^{\infty} x^{-2.25}dx = -\frac{(2)^{1.25}\,x^{-1.25}}{(0.8)(1.25)}\Bigg|_{10}^{\infty} = \left(\frac{2}{10}\right)^{1.25} = 0.1337$$

Solution 6.17

The mean is:

$$E[X] = 10\,\Gamma(5) = 10 \times 4! = 240$$

The variance is:

$$\text{var}[X] = E\left[X^2\right] - \left(E[X]\right)^2 = 10^2\,\Gamma(9) - (240)^2 = 100 \times 8! - (240)^2 = 3{,}974{,}400\,.$$

Solution 6.18

First we need the value of c. This can be found from:

$$1 = \int_0^{\infty} c\,x^2\,e^{-4x^3}\,dx$$

$$\Rightarrow \frac{1}{c} = \int_0^{\infty} x^2\,e^{-4x^3}\,dx$$

Now:

$$\int g'(x)\, e^{g(x)}\, dx = e^{g(x)} + k$$

where k is a constant.

So:

$$\frac{1}{c} = -\frac{1}{12}\int_0^\infty -12\,x^2\, e^{-4x^3}\, dx = -\left.\frac{e^{-4x^3}}{12}\right|_0^\infty = \frac{1}{12}$$

$$\Rightarrow c = 12$$

Now we need:

$$\Pr\left(X > 0.5\right) = \int_{0.5}^\infty 12\,x^2\, e^{-4x^3}\, dx = -e^{-4x^3}\Big|_{0.5}^\infty = e^{-4(0.5)^3} = e^{-0.5} = 0.6065$$

Solution 6.19

The median of X is $x_{0.5}$ where:

$$0.5 = \int_0^{x_{0.5}} f(x)\, dx = \int_0^{x_{0.5}} 7\left(1-x\right)^6\, dx = -\left(1-x\right)^7\Big|_0^{x_{0.5}} = 1-\left(1-x_{0.5}\right)^7$$

$$\Rightarrow 1 - x_{0.5} = \left(0.5\right)^{1/7} \Rightarrow x_{0.5} = 1-\left(0.5\right)^{1/7} = 0.0943$$

Solution 6.20

We first need to find c, using:

$$1 = \int_0^1 f(x)\, dx = c\int_0^1 \left(1-x\right)^8\, dx = -c\left.\frac{\left(1-x\right)^9}{9}\right|_0^1 = \frac{c}{9} \Rightarrow c = 9$$

The expected fraction of defective fuses is:

$$E\left[X\right] = \int_0^1 x\, f(x)\, dx = 9\int_0^1 x\left(1-x\right)^8\, dx$$

Integrating by parts:

$$E\left[X\right] = -x\left(1-x\right)^9\Big|_0^1 + \int_0^1 \left(1-x\right)^9\, dx = -\left.\frac{\left(1-x\right)^{10}}{10}\right|_0^1 = \frac{1}{10}$$

So we expect there to be 1,000 defective fuses in the batch of 10,000.