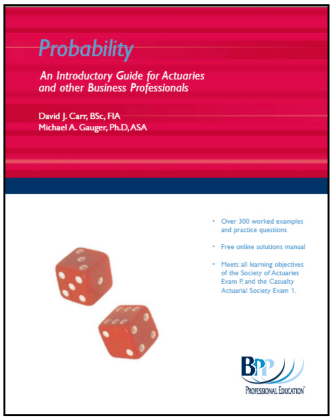

Probability

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Published by BPP Professional Education

Solutions to practice questions – Chapter 6

Solution 6.1

Let the losses due to storm, fire, and theft be denoted X_1, X_2 and X_3 respectively.

$$\begin{aligned}\Pr\left(\max\{X_1, X_2, X_3\} > 2.5\right) &= 1 - \Pr\left(\max\{X_1, X_2, X_3\} \le 2.5\right)\\ &= 1 - \Pr(X_1 \le 2.5) \times \Pr(X_2 \le 2.5) \times \Pr(X_3 \le 2.5)\end{aligned}$$

The distribution function for a uniform distribution with range $[a,b]$ is:

$$F(x) = \frac{x-a}{b-a}$$

So, we can calculate the probability as:

$$\begin{aligned}\Pr\left(\max\{X_1, X_2, X_3\} > 2.5\right) &= 1 - \Pr(X_1 \le 2.5) \times \Pr(X_2 \le 2.5) \times \Pr(X_3 \le 2.5)\\ &= 1 - (1)\left(\frac{2.5}{3}\right)\left(\frac{2.5}{4}\right) = 1 - 0.521 = 0.479\end{aligned}$$

Solution 6.2

We have:

$$E[X] = \frac{a+b}{2} = 1{,}879$$

$$\operatorname{var}(X) = \frac{(b-a)^2}{12} = 507 \quad \Rightarrow \quad b - a = \sqrt{12 \times 507} = 78$$

Hence:

$$a = 1{,}840 \qquad b = 1{,}918$$

Solution 6.3

Since we have formulas for $E[X]$ and $E[X^2]$, it is convenient to first calculate the second moment:

$$E\left[X^2\right] = \text{var}(X) + \left(E[X]\right)^2 = 18.75 + 2.5^2 = 25$$

Hence:

$$2.5 = E[X] = \frac{\theta}{\alpha-1}$$

$$25 = E\left[X^2\right] = \frac{\theta^2 2!}{(\alpha-1)(\alpha-2)}$$

We solve these simultaneous equations to find the parameters:

$$\frac{25}{2.5^2} = \frac{E[X^2]}{\left(E[X]\right)^2} = \frac{\theta^2 2!}{(\alpha-1)(\alpha-2)} \times \left(\frac{\alpha-1}{\theta}\right)^2 = \frac{2(\alpha-1)}{\alpha-2}$$

$$\Rightarrow \quad \alpha = 3, \ \theta = 5$$

Now it is easy to calculate the 90th percentile:

$$0.9 = F(x_{0.9}) = 1 - \left(\frac{\theta}{x_{0.9} + \theta}\right)^\alpha = 1 - \left(\frac{5}{x_{0.9} + 5}\right)^3$$

$$\Rightarrow \quad x_{0.9} = 5\left(0.1^{-1/3} - 1\right) = 5.772$$

Solution 6.4

If we notice that X follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 2{,}000$, the expected return is easily calculated using the general result as:

$$E[X] = \frac{\theta}{\alpha-1} = \frac{2{,}000}{3-1} = 1{,}000$$

Solution 6.5

We want to calculate the following:

$$\Pr\left(X < 2 \middle| X > 1.5\right) = \frac{\Pr(X < 2 \cap X > 1.5)}{\Pr(X > 1.5)} = \frac{\Pr(1.5 < X < 2)}{\Pr(X > 1.5)} = \frac{F(2) - F(1.5)}{1 - F(1.5)}$$

Note that the pdf is that of a one-parameter Pareto distribution with $\alpha = 3$ and $\theta = 1$, so the cdf is:

$$F(x) = 1 - x^{-3} \qquad \text{for } x > 1$$

Finally:

$$\Pr\left(X < 2 \middle| X > 1.5\right) = \frac{F(2) - F(1.5)}{1 - F(1.5)} = \frac{\left(1 - 2^{-3}\right) - \left(1 - 1.5^{-3}\right)}{1.5^{-3}} = 0.5781$$

Solution 6.6

The lifetime X follows a single-parameter Pareto distribution with $\alpha = 3$ and $\theta = 3$.

Recall that we can define $X = Y + \theta$, where Y follows a standard (two-parameter) Pareto distribution with $\alpha = 3$ and $\theta = 3$.

The expected value of X is given by:

$$E[X] = E[Y + \theta] = E[Y] + \theta = \frac{\theta}{\alpha - 1} + \theta = \frac{\alpha\theta}{\alpha - 1}$$

Hence, the expected value is:

$$E[X] = \frac{\alpha\theta}{\alpha - 1} = \frac{3 \times 3}{2} = 4.5$$

Solution 6.7

For the exponential distribution with mean 50, we have:

$$\Pr(X > x) = 1 - F(x) = e^{-x/50}$$

Hence:

$$\Pr\left(X > 200 \middle| X > 50\right) = \frac{\Pr(X > 200)}{\Pr(X > 50)} = \frac{e^{-200/50}}{e^{-50/50}} = \frac{e^{-4}}{e^{-1}} = e^{-3} = 0.0498$$

Alternatively, using the memoryless property we have:

$$\Pr\left(X > 200 \middle| X > 50\right) = \Pr(X > 150) = e^{-150/50} = e^{-3} = 0.0498$$

Solution 6.8

Let X be the random time (in days) until the high-risk driver has an accident. Since X follows an exponential distribution, the cdf is:

$$F(x) = 1 - e^{-x/\theta}$$

We are given that

$$0.30 = \Pr(X < 50) = 1 - e^{-50/\theta}$$

$$\Rightarrow \quad e^{-50/\theta} = 0.70$$

Hence:

$$\Pr(X < 80) = 1 - e^{-80/\theta} = 1 - \left(e^{-50/\theta}\right)^{8/5} = 1 - 0.70^{8/5} = 0.4349$$

Note: It is unnecessary to calculate the precise value of θ. If you did solve for θ, you should find that $\theta = 140.184$.

Solution 6.9

Let T denote the random time required to repair the machine. It is assumed to be exponentially distributed with mean $\theta_T = 2$. Let X denote the cost of replacement parts. It is assumed to be gamma distributed with parameters α and θ_X such that:

$$100 = E[X] = \alpha\theta_X$$

$$5{,}000 = \text{var}(X) = \alpha\theta_X^2$$

Solving these equations results in:

$$\alpha = 2 \qquad \theta_X = 50$$

We are asked to calculate the probability of the event $\{T > 3\} \cup \{X > 150\}$.

We can calculate the two components as follows:

$$\Pr(T > 3) = 1 - \Pr(T < 3) = e^{-3/2} = 0.2231$$

$$\Pr(X > 150) = 1 - \Pr(X < 150) = e^{-150/50}\left(1 + \frac{150}{50}\right) = 0.1991$$

By the additive probability law, we have:

$$\begin{aligned}\Pr(\{T>3\} \cup \{X>150\}) &= \Pr(T>3) + \Pr(X>150) - \Pr(\{T>3\} \cap \{X>150\}) \\ &= \Pr(T>3) + \Pr(X>150) - \Pr(T>3)\Pr(X>150) \\ &= 0.2231 + 0.1991 - 0.2231 \times 0.1991 = 0.3778\end{aligned}$$

Solution 6.10

We can calculate the required probability as follows:

$$\Pr(4<S<8) = \Pr(4<S<8 \cap N=0) + \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1)$$

If there are no claims (*ie* $N=0$), then the claim amount must be zero, so:

$$\Pr(4<S<8 \cap N=0) = 0$$

Therefore:

$$\Pr(4<S<8) = \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1)$$

So from basic laws of probability, we have:

$$\begin{aligned}\Pr(4<S<8) &= \Pr(4<S<8 \cap N=1) + \Pr(4<S<8 \cap N>1) \\ &= \Pr(4<S<8 \mid N=1) \times \Pr(N=1) + \Pr(4<S<8 \mid N>1) \times \Pr(N>1) \\ &= \underbrace{(F(8)-F(4))}_{\text{exp, } \theta=5} \times \tfrac{1}{3} + \underbrace{(F(8)-F(4))}_{\text{exp, } \theta=8} \times \tfrac{1}{6} \\ &= \left((1-e^{-8/5}) - (1-e^{-4/5})\right) \times \tfrac{1}{3} + \left((1-e^{-8/8}) - (1-e^{-4/8})\right) \times \tfrac{1}{6} \\ &= 0.1223\end{aligned}$$

Solution 6.11

The waiting time between accidents follows an exponential distribution with mean $\theta = 5/2 = 2.5$ days.

Hence:

$$\Pr(X > 3) = 1 - F(3) = e^{-3/2.5} = e^{-1.2} = 0.3012$$

Solution 6.12

From the form of the pdf, we can see that X follows a gamma distribution with $\alpha = 6$ and $\theta = 100$.

Hence:

$$E[X] = \alpha\theta = 6 \times 100 = 600$$

Solution 6.13

First, we'll determine the parameter values:

$$15 = E[X] = \alpha\theta$$

$$75 = \text{var}(X) = \alpha\theta^2$$

$$\Rightarrow \quad \theta = \frac{\text{var}(X)}{E[X]} = 5 \quad \Rightarrow \quad \alpha = 3$$

Since α is a positive integer, the cdf can be written as:

$$F(x) = 1 - e^{-x/5}\left(1 + \frac{x}{5} + \frac{x^2}{2!5^2}\right)$$

Finally:

$$\Pr(X > 30 \mid X > 15) = \frac{\Pr(X > 30)}{\Pr(X > 15)} = \frac{1 - F(30)}{1 - F(15)} = \frac{1 - 0.93803}{1 - 0.57681} = \frac{0.06197}{0.42319} = 0.1464$$

Solution 6.14

From the form of the moment generating function, we can see that X follows a gamma distribution with parameters $\alpha = 2$ and $\theta = 3$. Hence:

$$\text{var}(X) = \alpha\theta^2 = 2 \times 3^2 = 18$$

Solution 6.15

Differentiating the cumulant generating function, we have:

$$R_X(t) = \ln M_X(t) = \ln\left((1-\theta t)^{-\alpha}\right) = -\alpha \ln(1-\theta t)$$

$$R'_X(t) = \frac{\alpha\theta}{(1-\theta t)}$$

$$R''_X(t) = \frac{\alpha\theta^2}{(1-\theta t)^2} \quad \Rightarrow \quad R''_X(0) = \alpha\theta^2$$

$$R'''_X(t) = \frac{2\alpha\theta^3}{(1-\theta t)^3} \quad \Rightarrow \quad R'''_X(0) = 2\alpha\theta^3$$

Hence the skewness is:

$$\frac{E\left[(X-\mu)^3\right]}{\sigma^3} = \frac{R'''_X(0)}{\left(R''_X(0)\right)^{3/2}} = \frac{2\alpha\theta^3}{\alpha^{3/2}\theta^3} = \frac{2}{\sqrt{\alpha}}$$

Solution 6.16

Let X be the sum of the two sample values.

By the additive property, the sum of the two sample values follows a chi-square distribution with 2 degrees of freedom. A chi-square distribution with 2 degrees of freedom is equivalent to an exponential distribution with parameter $\theta = 2$.

Hence the required probability is:

$$\Pr(\text{Sample mean} < 1.15) = \Pr(X < 2.3) = F(2.3) = 1 - e^{-2.3/2} = 0.6834$$

Solution 6.17

With $\tau = 0.25$ and $\theta = 10$, we have:

$$E[X] = \theta\Gamma\left(1 + \frac{1}{\tau}\right) = 10\Gamma(5) = 10 \times 4! = 240$$

$$\text{var}(X) = \frac{\theta^2}{\tau}\left[2\Gamma(2/\tau) - \frac{1}{\tau}\left(\Gamma(1/\tau)\right)^2\right] = \frac{10^2}{0.25}\left[2\Gamma(8) - 4\left(\Gamma(4)\right)^2\right] = 3{,}974{,}400$$

Solution 6.18

The given pdf is for a Weibull distribution with $\tau = 3$ and $\theta = \sqrt[3]{1/4}$.

Using the general result for the cdf, we have:

$$F(x) = 1 - e^{-(x/\theta)^{\tau}} = 1 - e^{-4x^3}$$

$$\Rightarrow \Pr(X > 0.5) = 1 - F(0.5) = e^{-4\times 0.5^3} = e^{-0.5} = 0.6065$$

Solution 6.19

With $a = 1$ and $b = 7$, the pdf is:

$$f(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!}\, x^{a-1}\,(1-x)^{b-1} = \frac{7!}{6!}\,(1-x)^6 = 7(1-x)^6$$

Hence, the cdf is:

$$F(x) = \int_0^x f(y)\,dy = \int_0^x 7(1-y)^6\,dy = -(1-y)^7\Big|_0^x = 1-(1-x)^7$$

Now we can compute the median as:

$$0.5 = F(x_{0.5}) = 1-(1-x_{0.5})^7 \quad \Rightarrow \quad x_{0.5} = 1 - 0.5^{1/7} = 0.0943$$

Solution 6.20

The second moment is:

$$E\left[X^2\right] = \text{var}(X) + (E[X])^2 = \frac{1}{48} + \left(\frac{1}{4}\right)^2 = \frac{1}{12}$$

Solving the moment equations for the parameters, we have:

$$E[X] = \frac{a}{a+b} = \frac{1}{4} \quad \Rightarrow b = 3a$$

$$E\left[X^2\right] = \left(\frac{a}{a+b}\right)\left(\frac{a+1}{a+b+1}\right) = \frac{1}{12} \quad \Rightarrow \quad \frac{a+1}{a+b+1} = \frac{1}{3}$$

Substituting, we have:

$$\frac{a+1}{4a+1} = \frac{1}{3} = \frac{3}{9} \quad \Rightarrow \quad a = 2 \quad \Rightarrow \quad b = 6$$

