# 1

# Introduction to Probability Theory

## Overview

The concept of probability is commonly used in everyday life, and can be expressed in many ways. For example, there is a 50:50 chance of a "head" when a fair coin is tossed. There is a 1-in-6 chance of scoring a four when rolling a fair six-sided die. And a meteorologist may tell us that the chance of rain tomorrow is 80%.

In this introductory chapter, we'll define very precisely what we mean by a probability, and we'll study some important characteristics of probabilities. Throughout this book, we'll demonstrate the applications of probability theory in the areas of insurance and risk management. For example, we may wish to answer questions such as:

- What is the probability that a policyholder will file a claim in excess of $1,000 on his medical insurance policy this month?

- What is the probability that two machines fail on the same day?

- What is the probability that a policyholder with an automobile policy and a homeowners policy will renew at least one policy next year?

## 1.1   Set theory

Probability theory is founded on set theory. While we fully expect all readers to be familiar with basic set theory from high school, we will devote this initial section of the book to a short review of the main notation and concepts.

### Sets and elements

A **set** is a well-defined collection of **elements** or **members**.

If the element $x$ belongs to the set $A$, then we write $x \in A$. If the element $x$ does not belong to the set $A$, then we write $x \notin A$.

For example, if $A = \{1, 2, 3, 4\}$ then $2 \in A$ but $2.5 \notin A$ and $8 \notin A$.

The elements of a set may be **quantitative**, *eg* $A = \{0, 1\}$ or $B = \{3.42, 5.66, 8.17\}$, or **qualitative**, *eg* $C = \{\text{white, red, blue}\}$ or $D = \{\text{good, average, poor}\}$, or a combination.

A set may contain a finite or infinite number of elements. When a set contains a large number of elements, it may be impractical or impossible to list all of the elements using the notation above. For example, if $A$ is the set of all integers between 1 and 100 inclusive, then the set may be defined in any of the following ways:

$$A = \{\text{All integers between 1 and 100 inclusive}\}$$

$$A = \{1, 2, 3, \cdots, 99, 100\}$$

$$A = \{x : x \in \mathbb{Z}, 1 \leq x \leq 100\} \quad \text{where } \mathbb{Z} \text{ denotes the set of integers}$$

A set may also contain an uncountable number of elements. For example, if $A$ is the set of all real numbers greater than 5 but less than 75, we have:

$$A = \{x : x \in \mathbb{R}, 5 < x < 75\} \quad \text{where } \mathbb{R} \text{ denotes the set of real numbers}$$

### Subsets

The set $A$ is a **subset** of set $B$ if every element of set $A$ is also an element of set $B$.

If $A$ is a **subset** of $B$, then we write $A \subset B$.

If $A$ is not a **subset** of $B$, then we write $A \not\subset B$.

For example, if $A = \{1, 3, 5\}$, $B = \{5, 7\}$ and $C = \{1, 2, 3, 4, 5, 6\}$ then $A \subset C$ but $B \not\subset C$.

The definition leads easily to the following result:

$$A \subset B, B \subset A \Rightarrow A = B$$

### The universal set

In any particular situation it is useful to define the set of all possible elements of interest.

The set of all elements under consideration in any particular situation is called the **universal set**, which we will denote $S$.

For any set $A$, we have $A \subset S$.

For example, if we are interested in the score that might be obtained by rolling a fair die, the universal set would be $S = \{1, 2, 3, 4, 5, 6\}$, and we would restrict our consideration to subsets containing some or all of these six elements.

### The empty set

It is also useful to define a set with no elements.

The **empty set** or **null set** contains no elements, and is denoted $\varnothing$:

$$\varnothing = \{\ \}$$

The empty set is considered to be a subset of every other set, *ie*:

$$\varnothing \subset A \quad \text{for all sets } A$$

### Basic set operations

There are three basic set operations: intersection, union and complement.

The **intersection** of two sets $A$ and $B$ is the set of all elements that belong to both $A$ and $B$. It is denoted $A \cap B$ and is pronounced "$A$ intersection $B$."

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The **union** of two sets $A$ and $B$ is the set of all elements that belong to either $A$ or $B$ or both. It is denoted $A \cup B$ and is pronounced "$A$ union $B$."

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The **complement** of the set $A$ is the set of all elements that belong to the universal set $S$ but do not belong to set $A$. It is denoted $A'$.

$$A' = \{x : x \in S \text{ and } x \notin A\}$$

**Note:** The complement is sometimes denoted $\overline{A}$ or $A^c$.

For example, let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, and $B = \{4, 5, 6\}$. Then:

$$A \cap B = \{5\}$$
$$A \cup B = \{1, 3, 4, 5, 6\}$$
$$A' = \{2, 4, 6\}$$
$$B' = \{1, 2, 3\}$$

### Venn diagrams

A Venn diagram is a very useful graphical representation of sets. Skillful use of a Venn diagram can sometimes lead to a relatively simple solution to a problem that may look complicated at first sight.

In this book, the universal set $S$ will be represented by the interior of the rectangular area in the diagram. All subsets of $S$ will represented by the interior of other enclosed areas (usually circles) within $S$.

Here are some examples of Venn diagrams.  The shaded area in each diagram is described below each figure.



$S$



$A \subset S$



$A'$



$B \subset A$



$A \cap B$



$A \cup B$
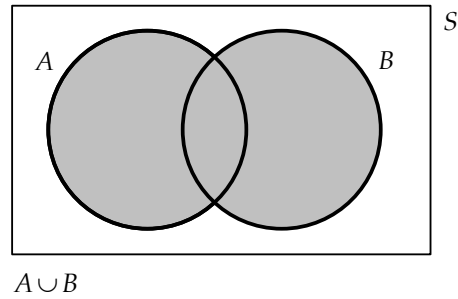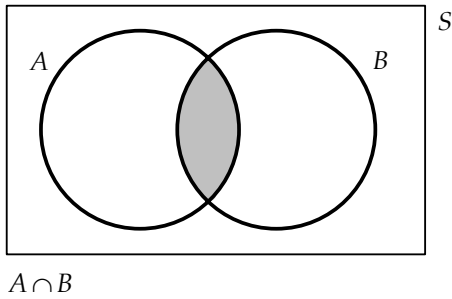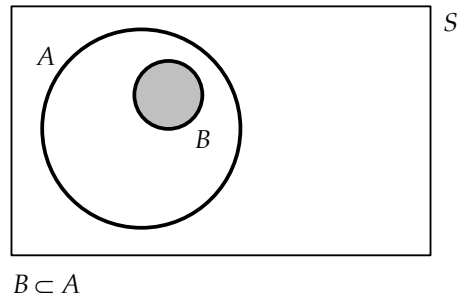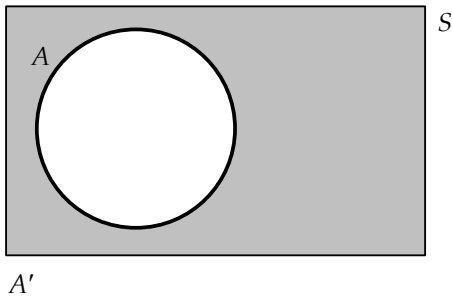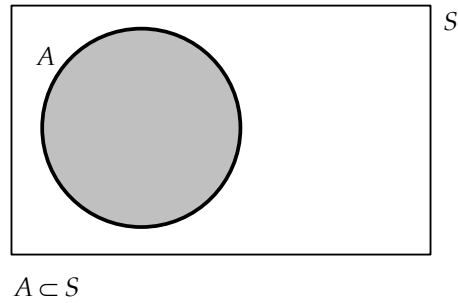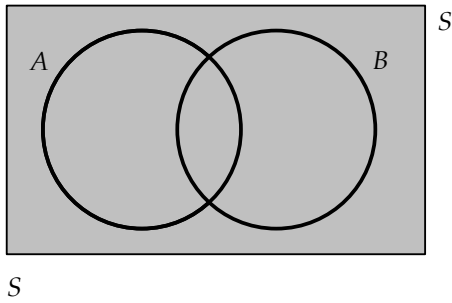
## *Basic laws of set operations*

The following laws are useful when working with set theory algebra:

The **associative laws** state that:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The **distributive laws** state that:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

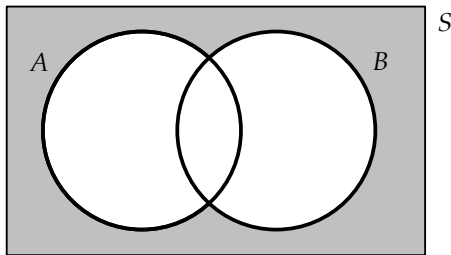$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

**De Morgan's laws** state that:
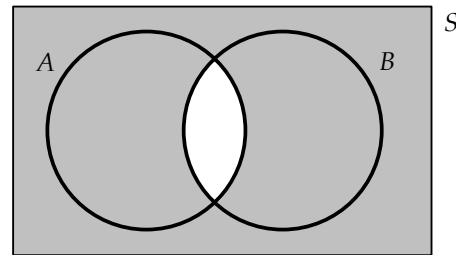
$$\left(A \cup B\right)' = A' \cap B'$$

$$\left(A \cap B\right)' = A' \cup B'$$

Given the review nature of this section, we will not prove these results, but each of these laws can be proved from first principles, or can be illustrated using a Venn diagram.

For example, De Morgan's laws can be illustrated as follows:



$$\left(A \cup B\right)' = A' \cap B' \qquad\qquad\qquad \left(A \cap B\right)' = A' \cup B'$$

## 1.2  Probability

We'll start with some basic definitions. These are based on the set theory concepts in Section 1.1 but have different nomenclature in probability theory, where we are interested in the possible outcomes of a random experiment.

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**Events**

The set of all possible outcomes of a random experiment is called the **sample space**, and is denoted $S$.

A subset of the sample space is called an **event**. An event is said to have occurred if the outcome of the experiment is in the corresponding subset.

Events $A_1, A_2, \cdots$ are **mutually exclusive** or **disjoint** if:

$$A_i \cap A_j = \varnothing \text{ whenever } i \neq j$$

*ie* mutually exclusive events cannot occur simultaneously.

Events $A_1, A_2, \cdots$ are **exhaustive** if:

$$A_1 \cup A_2 \cup \cdots = S$$

*ie* the events $A_1, A_2, \cdots$ cover all the possible outcomes.

If events $A_1, A_2, \cdots$ are mutually exclusive and exhaustive, then events $A_1, A_2, \cdots$ are said to form a **partition** of the sample space.

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For example, if the random experiment is the score obtained by rolling a fair die, the sample space is $S = \{1,2,3,4,5,6\}$. We can define the following events:

$$A_1 = \{\text{Score is odd number}\} = \{1,3,5\}$$
$$A_2 = \{\text{Score is even number}\} = \{2,4,6\}$$
$$A_3 = \{\text{Score is low}\} = \{1,2,3\}$$
$$A_4 = \{\text{Score is high}\} = \{4,5,6\}$$
$$A_5 = \{\text{Roll a 6}\} = \{6\}$$

Events $A_1$ and $A_2$ are mutually exclusive and exhaustive, so they form a partition of the sample space. The score must be either odd or even, but cannot be both.

Events $A_3$ and $A_4$ are also mutually exclusive and exhaustive, so they too form a partition of the sample space. The score must be either low or high, but cannot be both.

Events $A_3$ and $A_5$ are mutually exclusive, since the score cannot both be low and a 6. These events are not exhaustive, since they do not cover the possible outcomes of rolling a 4 or a 5.

Events $A_2$ and $A_4$ are neither mutually exclusive (a score of 4 or 6 is both even and high), nor exhaustive (they do not cover the possible outcomes of rolling a 1 or a 3).

Probability is a numerical measure of the likelihood of an event occurring. It is defined using a function that assigns a value to each subset of the sample space, subject to certain consistency conditions or axioms.

Probability is defined using the following axioms.

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**Axioms of probability**

The probability of event $A$ in the sample space $S$, denoted $\Pr(A)$, is a real number that satisfies the following:

(1)        $\Pr(A) \geq 0$  for any event $A$

(2)        $\Pr(S) = 1$

(3)        If $A_1, A_2, \cdots$ are mutually exclusive events, then:
$$\Pr(A_1 \cup A_2 \cup \cdots) = \Pr(A_1) + \Pr(A_2) + \cdots$$

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So, a probability is a real number between 0 and 1 (where a value of 0 represents the probability that an impossible event occurs, and a value of 1 represents the probability that a certain event occurs). We can prove this and other important relationships from these axioms.

### Theorem 1.1

$$\Pr(\varnothing) = 0$$

### Proof

Events $A$ and $\varnothing$ are mutually exclusive. From axiom (3), we have $\Pr(A \cup \varnothing) = \Pr(A) + \Pr(\varnothing)$ but since $A \cup \varnothing = A$, we have:

$$\Pr(A) = \Pr(A) + \Pr(\varnothing) \quad \Rightarrow \quad \Pr(\varnothing) = 0 \qquad\qquad \square$$

### *Theorem 1.2*

For any event $A$ :

$$\Pr(A') = 1 - \Pr(A)$$

### *Proof*

Events $A$ and $A'$ are mutually exclusive. From axiom (3), we have $\Pr(A \cup A') = \Pr(A) + \Pr(A')$ but since $A \cup A' = S$ , we have:

$$\Pr(S) = \Pr(A) + \Pr(A') \;\Rightarrow\; 1 = \Pr(A) + \Pr(A') \;\Rightarrow\; \Pr(A') = 1 - \Pr(A) \qquad \Box$$

### *Theorem 1.3*

For any event $A$ :

$$0 \le \Pr(A) \le 1$$

### *Proof*

From axiom (1), we know that $\Pr(A) \ge 0$ , and similarly $\Pr(A') \ge 0$ . From Theorem 1.2, we have:

$$\Pr(A) + \Pr(A') = 1$$

Since each quantity is non-negative, this implies that $0 \le \Pr(A) \le 1$ for any event $A$ . $\qquad \Box$

### *Theorem 1.4  (The additive probability law)*

For any two events $A$ and $B$ :

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

### *Proof*

Axiom (3) relates to mutually exclusive events, but events $A$ and $B$ do not necessarily satisfy this requirement. We proceed by expressing $A \cup B$ as the union of three mutually exclusive events (this relationship is simple to verify using a Venn diagram):

$$A \cup B = (A \cap B') \cup (A' \cap B) \cup (A \cap B)$$
$$\Rightarrow \Pr(A \cup B) = \Pr(A \cap B') + \Pr(A' \cap B) + \Pr(A \cap B)$$

We can also express events $A$ and $B$ as the union of mutually exclusive events:

$$A = (A \cap B) \cup (A \cap B') \quad \Rightarrow \quad \Pr(A \cap B') = \Pr(A) - \Pr(A \cap B)$$
$$B = (A \cap B) \cup (A' \cap B) \quad \Rightarrow \quad \Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B)$$

Substituting into the main formula, we have:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \qquad \Box$$

This result has an important intuitive interpretation: when we consider the probability of the occurrence of event $A$ **or** event $B$, we must not double count those outcomes for which events $A$ **and** $B$ both occur. Since the values of $\Pr(A)$ and $\Pr(B)$ both include $\Pr(A \cap B)$, we deduct $\Pr(A \cap B)$ in order to avoid counting these outcomes twice. Also, note that this rule reduces to Axiom (3) when events $A$ and $B$ are mutually exclusive.

The additive law can be extended to cover any three events, $A$, $B$, and $C$:

$$\Pr(A \cup B \cup C)$$
$$= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$$

## *Theorem 1.5   (The law of total probability)*

If the events $A_1, A_2, \cdots$ are mutually exclusive and exhaustive, then:

$$\Pr(B) = \Pr(B \cap A_1) + \Pr(B \cap A_2) + \cdots$$

## *Proof*

First, we will show that events $B \cap A_1, B \cap A_2, \cdots$ are mutually exclusive.

Since events $A_1, A_2, \cdots$ are mutually exclusive, $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Using the associative laws, we have:

$$(B \cap A_i) \cap (B \cap A_j) = B \cap (A_i \cap A_j) = B \cap \varnothing = \varnothing \quad \text{whenever } i \neq j$$

Next, we show that $B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots$

$$(B \cap A_1) \cup (B \cap A_2) \cup \cdots$$
$$= B \cap (A_1 \cup A_2 \cup \cdots) \qquad \text{(by the distributive laws)}$$
$$= B \cap S = B$$

Now it is a simple matter to use axiom (3) to provide the required result:

$$\Pr(B) = \Pr\big((B \cap A_1) \cup (B \cap A_2) \cup \cdots\big)$$
$$= \Pr(B \cap A_1) + \Pr(B \cap A_2) + \cdots$$

$\square$

Let's summarize these results and then work through some numerical examples.

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**Further properties of probability**

(1)      $\Pr(\varnothing) = 0$

(2)      For any event $A$:

$$\Pr(A') = 1 - \Pr(A)$$

(3)      For any event $A$:

$$0 \leq \Pr(A) \leq 1$$

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(4)       For any two events $A$ and $B$:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

(5)       If the events $A_1, A_2, \cdots$ are mutually exclusive and exhaustive, then:

$$\Pr(B) = \Pr(B \cap A_1) + \Pr(B \cap A_2) + \cdots$$

## *Example 1.1*

A manufacturing company operates two machines. The probability that the first machine breaks down in the next week is 0.14. The probability that the second machine breaks down in the next week is 0.17. The probability that both machines break down in the next week is 0.06. Calculate:

(i)        the probability that the first machine does not break down in the next week

(ii)      the probability that at least one machine breaks down in the next week

(iii)     the probability that exactly one machine breaks down in the next week.

## *Solution*

(i)       Let $A = \{\text{First machine breaks down}\}$ and $B = \{\text{Second machine breaks down}\}$.

From the question we have:

$$\Pr(A) = 0.14, \quad \Pr(B) = 0.17, \quad \Pr(A \cap B) = 0.06$$

The probability that the first machine does not break down in the next week is:

$$\Pr(A') = 1 - \Pr(A) = 1 - 0.14 = 0.86$$

(ii)     The probability that at least one machine breaks down in the next week is:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.14 + 0.17 - 0.06 = 0.25$$

(iii)    This final calculation is the most difficult.

The required event is:

$$(\text{First machine fails} \cap \text{Second machine does not fail})$$
$$\cup (\text{First machine does not fail} \cap \text{Second machine fails}) = (A \cap B') \cup (A' \cap B)$$

Since these two outcomes are mutually exclusive, the required probability is:

$$\Pr(A \cap B') + \Pr(A' \cap B)$$

Using basic set theory, we have:

$$A = (A \cap B) \cup (A \cap B')$$
$$\Rightarrow \Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$$
$$\Rightarrow \Pr(A \cap B') = \Pr(A) - \Pr(A \cap B) = 0.14 - 0.06 = 0.08$$

and similarly:

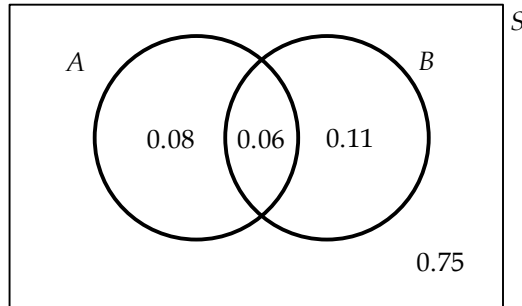$$B = (A \cap B) \cup (A' \cap B)$$
$$\Rightarrow \Pr(A' \cap B) = \Pr(B) - \Pr(A \cap B) = 0.17 - 0.06 = 0.11$$

Finally, the required probability is:

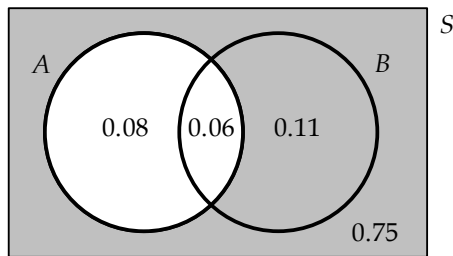$$\Pr(A \cap B') + \Pr(A' \cap B) = 0.08 + 0.11 = 0.19 \qquad \blacklozenge\blacklozenge$$

We can also answer this question using a Venn diagram. The probability of each region can be calculated easily from the information in the question:



For part (a), we need to consider the shaded area in the following Venn diagram:



So, the required probability is:

$$0.11 + 0.75 = 0.86$$

For part (b), we need to consider the shaded area in the following Venn diagram:
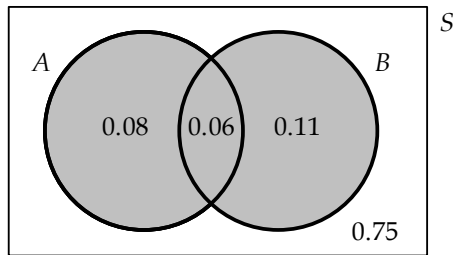


So, the required probability is:

$$0.08 + 0.06 + 0.11 = 0.25$$

For part (c), we need to consider the shaded area in the following Venn diagram:
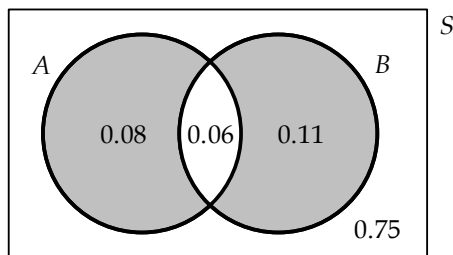


So, the required probability is:

$$0.08 + 0.11 = 0.19$$

## Example 1.2

The probability that a policyholder will make a claim on his homeowners insurance policy during the next year is 0.44. The probability that a policyholder will make a claim on his automobile insurance policy during the next year is 0.36. The probability that a policyholder will make a claim on both his homeowners insurance policy and his automobile insurance policy during the next year is 0.21.

Calculate the probability that a randomly selected policyholder does not make a claim on either his homeowners insurance policy or his automobile insurance policy during the next year.

### Solution

Let $A = \{$Claims on homeowners policy$\}$ and $B = \{$Claims on automobile policy$\}$.

From the question we have:

$$\Pr(A) = 0.44, \quad \Pr(B) = 0.36, \quad \Pr(A \cap B) = 0.21$$

Hence, the probability that a claim is made on either policy is:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 0.44 + 0.36 - 0.21 = 0.59$$

and the probability that a claim is made on neither policy is:

$$\Pr((A \cup B)') = 1 - \Pr(A \cup B) = 1 - 0.59 = 0.41 \qquad\qquad\qquad \blacklozenge\blacklozenge$$

We can also answer this question using a Venn diagram. The probability of each region can be calculated easily from the information in the question:
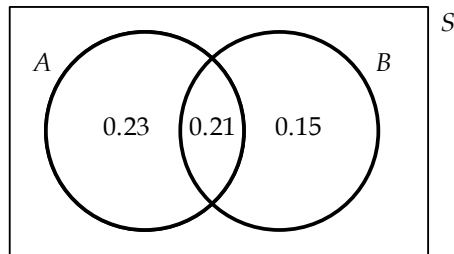


We need to calculate the probability of lying outside the circles. This is:

$$1 - 0.23 - 0.21 - 0.15 = 0.41$$

## Example 1.3

Among a large group of patients suffering from Systemic Lupus Erythematosus (SLE), it is found that 26% suffer from both arthritis and kidney problems, whereas 8% suffer from neither of these. The probability that a patient visits suffers from kidney problems exceeds by 0.16 the probability that a patient suffers from arthritis. Determine the probability that a randomly chosen member of this group suffers from arthritis.

### Solution

Let $A = \{$Suffers from arthritis$\}$ and $B = \{$Suffers from kidney problems$\}$.

From the question we have:

$$\Pr(A \cap B) = 0.26, \quad \Pr((A \cup B)') = 0.08, \quad \Pr(B) = \Pr(A) + 0.16$$

Using the property $\Pr(A') = 1 - \Pr(A)$, we have:

$$\Pr(A \cup B) = 1 - \Pr\big((A \cup B)'\big) = 1 - 0.08 = 0.92$$

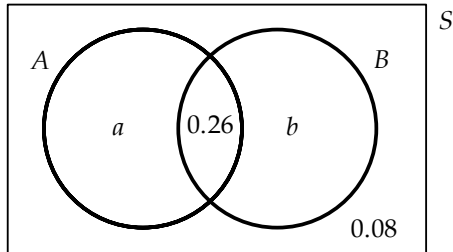And using the property $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, we have:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$\Rightarrow 0.92 = \Pr(A) + \big(\Pr(A) + 0.16\big) - 0.26$$
$$\Rightarrow \Pr(A) = 0.51 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \blacklozenge\blacklozenge$$

We can also answer this question using a Venn diagram. The probability of some regions can be filled in from the information in the question:



From the information in the question:

$$0.26 + b = 0.16 + (a + 0.26) \quad \Rightarrow \quad b - a = 0.16$$
$$a + 0.26 + b + 0.08 = 1 \quad \Rightarrow \quad b + a = 0.66$$

Subtracting the two equations, we get $2a = 0.5 \quad \Rightarrow \quad a = 0.25$, so that $\Pr(A) = 0.25 + 0.26 = 0.51$

## *Example 1.4*

A survey of a group of college students finds that:

- 64% eat Chinese food
- 51% eat Indian food
- 48% eat Thai food
- 31% eat Chinese and Indian food
- 27% eat Indian and Thai food
- 34% eat Chinese and Thai food
- 23% eat Chinese, Indian and Thai food.

Calculate the probability that a randomly selected student eats Chinese food but does not eat Indian or Thai food.

## *Solution*

This question is most easily solved using a Venn diagram.

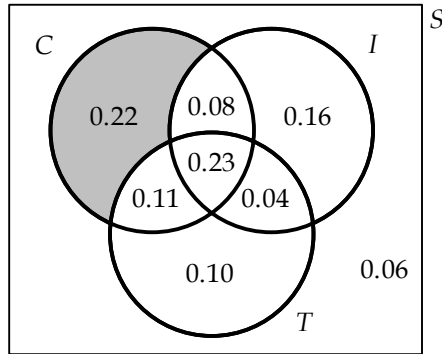From the question, $\Pr(C \cap I \cap T) = 0.23$.

Since $\Pr(C \cap I) = 0.31$, we have:

$$\Pr(C \cap I \cap T') = \Pr(C \cap I) - \Pr(C \cap I \cap T) = 0.31 - 0.23 = 0.08$$

Similarly, since $\Pr(C \cap T) = 0.34$, we have:

$$\begin{aligned}
\Pr(C \cap I' \cap T) &= \Pr(C \cap T) - \Pr(C \cap I \cap T) \\
&= 0.34 - 0.23 \\
&= 0.11
\end{aligned}$$

The other probabilities can be calculated in a similar fashion. From the diagram, we can see that the required probability (the shaded region) is 0.22.



The question can also be solved algebraically:

$$\begin{aligned}
\Pr(C \cap I' \cap T') &= \Pr(C) - \Pr(C \cap I' \cap T) - \Pr(C \cap I \cap T') - \Pr(C \cap I \cap T) \\
&= \Pr(C) - \big(\Pr(C \cap T) - \Pr(C \cap I \cap T)\big) \\
&\quad - \big(\Pr(C \cap I) - \Pr(C \cap I \cap T)\big) - \Pr(C \cap I \cap T) \\
&= \Pr(C) - \Pr(C \cap T) - \Pr(C \cap I) + \Pr(C \cap I \cap T) \\
&= 0.64 - 0.34 - 0.31 + 0.23 = 0.22 \qquad\qquad\qquad\qquad \blacklozenge\blacklozenge
\end{aligned}$$

## 1.3   *Equally likely events*

In many cases, it may seem appropriate to assume that each possible outcome of an experiment is equally likely to occur. For example, the probability of observing a head when a fair coin is tossed is equal to the probability of observing a tail. And there is an equal probability of scoring 1, 2, 3, 4, 5 or 6 when rolling a fair die.

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**Equally likely events**

The sample space for an experiment comprises $n$ possible outcomes, $S = \{o_1, o_2, \cdots, o_n\}$.

The $n$ possible outcomes $\{o_1, o_2, \cdots, o_n\}$ are **equally likely** (or **equiprobable**) if:

$$\Pr(\{o_i\}) = \frac{1}{n} \qquad \text{for } i = 1, 2, \cdots, n$$

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For example, a date in January is chosen at random. There are 31 days in January, so there are 31 possible outcomes and:

$$\Pr\left(\text{Date chosen is January 1}\right) = \frac{1}{31}$$

$$\Pr\left(\text{Date chosen is January 2}\right) = \frac{1}{31}$$

$$\vdots$$

$$\Pr\left(\text{Date chosen is January 31}\right) = \frac{1}{31}$$

---

**Compound events**

A **compound event** corresponds to a set of possible outcomes $E \subset S$, with probability:

$$\Pr(E) = \sum_{o_i \in E} \Pr(\{o_i\})$$

When the $n$ possible outcomes of an experiment $\{o_1, o_2, \cdots, o_n\}$ are equally likely, the probability of a compound event $E \subset S$ is given by:

$$\frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } S} = \frac{\text{Number of outcomes in } E}{n}$$

---

For example, continuing our example of selecting at random a date in January, we have:

$$\Pr\left(\text{Date chosen is before January 13}\right) = \frac{12}{31}$$

$$\Pr\left(\text{Date chosen is before January 7 or after January 27}\right) = \frac{6+4}{31} = \frac{10}{31}$$

$$\Pr\left(\text{Date chosen is an odd number}\right) = \frac{16}{31}$$

### Example 1.5

An insurance company has 12,568 automobile policyholders, of which 3,088 live in California. What is the probability that a randomly selected automobile policyholder lives in California?

### Solution

The phrase "randomly selected" is commonly used to indicate that each outcome (in this case that a given policyholder is selected) is equally likely.

So, if each policyholder is equally likely to be selected, then the probability that a randomly selected policyholder lives in California is:

$$\frac{\text{Number of policyholders who live in California}}{\text{Total number of policyholders}} = \frac{3,088}{12,568} = 0.2457 \qquad \blacklozenge\blacklozenge$$

### Example 1.6

An urn contains 4 red balls, 6 yellow balls and 5 blue balls. Calculate the probability that a ball drawn at random is red or blue.

### Solution

Each ball is equally likely to be drawn. The probability that a randomly selected ball is red or blue is:

$$\frac{\text{Number of balls that are red or blue}}{\text{Total number of balls}} = \frac{4+5}{4+6+5} = \frac{9}{15} = 0.6 \qquad \blacklozenge\blacklozenge$$

### Example 1.7

A card is drawn at random from a standard pack of playing cards. Calculate the probability that the card drawn is a club or a 10.

### Solution

Each of the 52 cards is equally likely to be drawn.

In total, 16 cards satisfy the condition of being a club or a 10 – these are the 13 clubs (including the 10 of clubs), the 10 of hearts, the 10 of diamonds, and the 10 of spades.

So, the probability that a randomly selected card is a club or a 10 is:

$$\frac{\text{Number of cards that are a club or a 10}}{\text{Total number of cards}} = \frac{16}{52} = 0.3077 \qquad \blacklozenge\blacklozenge$$

**Note:** In these examples, it is easy to count the number of possible outcomes within the specified event, and the total number of possible outcomes. In other situations, such calculations can be surprisingly complicated. We'll study more complicated counting problems in Chapter 2.

## 1.4   Independent events

The concept of statistical independence corresponds to the intuitive idea that two events are independent if the probability that the first event occurs is not influenced by the occurrence (or non-occurrence) of the second event, and vice versa. For example, it is intuitive that the scores obtained by rolling two fair dice should not be related in any predictable way.

### Independence of two events

The formal definition of independence is as follows.

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**Independent events**

Events $A$ and $B$ are **independent** if and only if:

$$\Pr(A \cap B) = \Pr(A)\Pr(B)$$

---

For example, two fair coins are tossed. The sample space is $S = \{HH, HT, TH, TT\}$, where $H$ represents a head and $T$ represents a tail. Each outcome is equally likely.

Let $A = \{$First coin is a head$\}$ and $B = \{$Second coin is a head$\}$. The probability that both coins are a head is $\Pr(A \cap B) = 1/4$, since there are four equally likely outcomes, and there is only one

way in which both coins can be a head. The probability that the first coin is a head is $\Pr(A) = 1/2$, and the probability that the second coin is a head is $\Pr(B) = 1/2$.

The two events $A$ and $B$ are independent since:

$$\Pr(A)\Pr(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \Pr(A \cap B)$$

In another experiment, two fair dice are rolled. There are 36 equally likely outcomes:

(1,1)   (1,2)   (1,3)   (1,4)   (1,5)   (1,6)

(2,1)   (2,2)   (2,3)   (2,4)   (2,5)   (2,6)

(3,1)   (3,2)   (3,3)   (3,4)   (3,5)   (3,6)

(4,1)   (4,2)   (4,3)   (4,4)   (4,5)   (4,6)

(5,1)   (5,2)   (5,3)   (5,4)   (5,5)   (5,6)

(6,1)   (6,2)   (6,3)   (6,4)   (6,5)   (6,6)

Let $A = \{$First die is a 2 or a 4$\}$ and $B = \{$Second die is a 1 or a 3 or a 5$\}$. Then $\Pr(A) = 2/6 = 1/3$ and $\Pr(B) = 3/6 = 1/2$.

The event $A \cap B$ contains 6 outcomes: (2,1), (2,3), (2,5), (4,1), (4,3), and (4,5). So:

$$\Pr(A \cap B) = 6/36 = 1/6$$

The two events $A$ and $B$ are independent since:

$$\Pr(A)\Pr(B) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = \Pr(A \cap B)$$

When two events $A$ and $B$ are not independent, they are said to be **dependent** events.

To illustrate this, let's consider an experiment in which one fair die is rolled and the score is noted. Let $A = \{$Score is an odd number$\} = \{1, 3, 5\}$ and $B = \{$Score is low$\} = \{1, 2, 3\}$. Clearly, $\Pr(A) = 1/2$ and $\Pr(B) = 1/2$.

The event $A \cap B = \{1, 3\}$, so $\Pr(A \cap B) = 2/6 = 1/3$.

We can see that events $A$ and $B$ are dependent since:

$$\Pr(A)\Pr(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq \Pr(A \cap B)$$

## *Independence of three or more events*

So far we've considered the independence (or otherwise) of two events. We can extend this concept to three or more events.

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**Pairwise independence and mutual independence**

Events $A$, $B$ and $C$ are **pairwise independent** if and only if:

$$\Pr(A \cap B) = \Pr(A)\Pr(B), \ \Pr(A \cap C) = \Pr(A)\Pr(C) \ \textbf{and} \ \Pr(B \cap C) = \Pr(B)\Pr(C)$$

Events $A$, $B$ and $C$ are **mutually independent** if and only if these events are pairwise independent and:

$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$$

---

To illustrate this idea, let's consider an experiment in which a computer randomly selects an integer between 1 and 9 (inclusive). So, $S = \{1,2,3,4,5,6,7,8,9\}$.

Let $A = \{1,2,3\}$, $B = \{2,5,8\}$, and $C = \{2,4,6\}$.

The events $A$, $B$ and $C$ are pairwise independent because:

$$\Pr(A)\Pr(B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} = \Pr(A \cap B)$$

$$\Pr(A)\Pr(C) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} = \Pr(A \cap C)$$

$$\Pr(B)\Pr(C) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} = \Pr(B \cap C)$$

However, events $A$, $B$ and $C$ are not mutually independent since:

$$\Pr(A)\Pr(B)\Pr(C) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27} \neq \Pr(A \cap B \cap C) = \frac{1}{9}$$

## *Other properties of independent events*

It is intuitive that if event $A$ is independent of the occurrence of event $B$, then it should also be independent of the non-occurrence of event $B$. In other words, whether or not event $B$ happens, the probability of event $A$ is unaffected.

This is restated formally as follows:

If events $A$ and $B$ are independent, then:

- events $A$ and $B'$ are independent, *ie* $\Pr(A \cap B') = \Pr(A)\Pr(B')$

- events $A'$ and $B$ are independent, *ie* $\Pr(A' \cap B) = \Pr(A')\Pr(B)$

- events $A'$ and $B'$ are independent, *ie* $\Pr(A' \cap B') = \Pr(A')\Pr(B')$.

Let us consider again the previous example in which a computer randomly selects an integer between 1 and 9. It is straightforward to confirm that:

$$\Pr(A)\Pr(B') = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9} = \Pr(A \cap B')$$

$$\Pr(A')\Pr(B) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9} = \Pr(A' \cap B)$$

$$\Pr(A')\Pr(B') = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9} = \Pr(A' \cap B')$$

## *Example 1.8*

There are two urns. The first urn contains 4 red balls and 6 blue balls, and the second urn contains 7 red balls and 3 blue balls. One ball is drawn at random from the first urn and a second ball is drawn at random from the second urn.

Calculate the probability that both of the balls drawn are red.

### Solution

Let $A = \{$Ball drawn from first urn is red$\}$ and $B = \{$Ball drawn from second urn is red$\}$. Then:

$$\Pr(A) = \frac{4}{10}, \quad \Pr(B) = \frac{7}{10}$$

Since the drawings are independent, the probability that both of the balls drawn are red is:

$$\Pr(A \cap B) = \Pr(A)\Pr(B) = \frac{4}{10} \times \frac{7}{10} = \frac{28}{100} = 0.28 \qquad\qquad \blacklozenge\blacklozenge$$

### Example 1.9

A game is played by rolling a fair die and drawing a card at random from a standard pack of playing cards. Calculate:

(i)        the probability of scoring a 6 with a die and drawing the ace of spades

(ii)       the probability of scoring more than a 4 with the die and drawing a club

(iii)      the probability that the score on the die is equal to the number on the card (where an ace is equal to 1).

### Solution

(i)        The two events (rolling the die, and drawing the card) are independent. So, the probability of scoring a 6 with a die and drawing the ace of spades is:

$$\Pr(6 \text{ with die} \cap \text{ace of spades}) = \Pr(6 \text{ with die})\Pr(\text{ace of spades})$$

$$= \frac{1}{6} \times \frac{1}{52} = 0.0032$$

(ii)       The probability of scoring more than a 4 with the die and drawing a club is:

$$\Pr(> 4 \text{ with die} \cap \text{club}) = \Pr(> 4 \text{ with die})\Pr(\text{club}) = \frac{2}{6} \times \frac{13}{52} = \frac{1}{3} \times \frac{1}{4} = 0.0833$$

(iii)      The probability that the score on the die is equal to the number on the card is:

$$\Pr(\text{Score on die} = \text{Number on card})$$

$$= \sum_{i=1}^{6} \Pr(\text{Score } i \text{ on die} \cap \text{Number } i \text{ on card})$$

$$= \sum_{i=1}^{6} \Pr(\text{Score } i \text{ on die})\Pr(\text{Number } i \text{ on card})$$

$$= \sum_{i=1}^{6} \left( \frac{1}{6} \times \frac{4}{52} \right) = \frac{1}{13} = 0.0769 \qquad\qquad \blacklozenge\blacklozenge$$

### Example 1.10

A company employs two warehouse managers: Mr Allen and Ms Barclay. The probability that Mr Allen is unable to work due to illness is 0.04, and the probability that Ms Barclay is unable to work due to illness is 0.03. Assuming that the health statuses of the two managers are independent, calculate the probability that both managers are able to work.

### Solution

Let $A = \{$Mr Allen is unable to work$\}$ and $B = \{$Ms Barclay is unable to work$\}$. Then:

$$\Pr(A) = 0.04, \quad \Pr(B) = 0.03$$

Since events $A$ and $B$ are independent, we have:

$$\Pr(A' \cap B') = \Pr(A')\Pr(B') = \big(1 - \Pr(A)\big)\big(1 - \Pr(B)\big)$$

$$= (1 - 0.04) \times (1 - 0.03) = 0.96 \times 0.97 = 0.9312 \qquad \blacklozenge\blacklozenge$$

### Example 1.11

A hydraulic press produces machine parts. There is a probability of 0.06 that a machine part will be faulty. The quality of each machine part is independent of the quality of any other part. Calculate:

(i)    the probability that none of the first five machine parts is faulty

(ii)   the probability that at least one of the first ten machine parts is faulty.

### Solution

(i)    Let $A_i = \{$Machine part $i$ is not faulty$\}$. Then $\Pr(A_i) = 1 - \Pr(A_i') = 1 - 0.06 = 0.94$.

Since $A_1, A_2, \cdots, A_5$ are independent, we have:

$$\Pr(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \prod_{i=1}^{5} \Pr(A_i) = 0.94^5 = 0.7339$$

(ii)   The probability that at least one of the first ten machine parts is faulty is:

$$1 - \Pr(A_1 \cap A_2 \cap \cdots \cap A_{10}) = 1 - \prod_{i=1}^{10} \Pr(A_i) = 1 - 0.94^{10} = 0.4614 \qquad \blacklozenge\blacklozenge$$

### Example 1.12

A computer manufacturer provides a one-year warranty with all new computers. The number of computers that are repaired under warranty due to faults with the hard drive is 65% of the total number of repairs. The number of computers repaired under warranty due to faults with neither the monitor nor the hard drive is 21% of the total number of repairs.

The occurrence of faults with the monitor is independent of the occurrence of faults with the hard drive. Calculate the probability that a randomly selected computer that is repaired under warranty was repaired due to faults with both the monitor and the hard drive.

### Solution

Let $A = \{$Faulty hard drive$\}$ and $B = \{$Faulty monitor$\}$. Then:

$$\Pr(A) = 0.65, \quad \Pr(A' \cap B') = 0.21$$

Since events $A$ and $B$ are independent, we have:

$$Pr(A' \cap B') = Pr(A')Pr(B') = (1 - Pr(A))(1 - Pr(B))$$
$$\Rightarrow 0.21 = (1 - 0.65)(1 - Pr(B))$$
$$\Rightarrow Pr(B) = 0.4$$

And finally:

$$Pr(A \cap B) = Pr(A)Pr(B) = 0.65 \times 0.4 = 0.26 \qquad\qquad \blacklozenge\blacklozenge$$

## Chapter 1 Practice Questions

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**Free online solutions manual**

You can download detailed worked solutions to every practice question in this book free of charge from the BPP Professional Education website at **www.bpptraining.com.** Select support for the SOA/CAS exams and click on the Probability (P) home page. You'll also find other useful study resources here.

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### Question 1.1

Let $A = \{2,3,5,7\}$, let $B = \{1,3,5,7,9\}$, and let the universal set $S = \{1,2,3,\cdots,9,10\}$. Determine:

(i)     $A \cap B$

(ii)    $A \cup B'$

(iii)   $(A \cup B)'$

### Question 1.2

Draw the sets $A$, $B$ and $S$ from Question 1.1 in a Venn diagram.

### Question 1.3

If $\Pr(A) = 0.4$, $\Pr(B) = 0.3$, and $\Pr(A \cap B) = 0.15$, calculate $\Pr(A \cup B)$.

### Question 1.4

If $\Pr(A) = \Pr((A \cup B)')$ and $\Pr(A \cup B) = 1.5\Pr(A)$, calculate $\Pr(A)$.

### Question 1.5

If $\Pr(A) = \Pr(B \cap A')$, $\Pr(A \cap B) = 0.1$, and $\Pr(A \cup B) = 0.8$, calculate $\Pr(B')$.

### Question 1.6

The events $A_1, A_2, A_3$ are mutually exclusive and exhaustive, and:

$$\Pr(A_2 \cap B) = 2 \times \Pr(A_1 \cap B) \qquad \Pr(A_3 \cap B) = 2 \times \Pr(A_2 \cap B) \qquad \Pr(B') = 0.3$$

Calculate $\Pr(A_1 \cap B)$.

### Question 1.7                                                     *SOA/CAS*

You are given that $\Pr(A \cup B) = 0.7$ and $\Pr(A \cup B') = 0.9$. Determine $\Pr(A)$.

## Question 1.8

The probability that a policyholder will make a claim on his homeowner's insurance policy during the next year is 0.46. The probability that a policyholder will make a claim on his automobile insurance policy during the next year is 0.32. The probability that a randomly selected policyholder does not make a claim on either his homeowner's insurance policy or his automobile insurance policy during the next year is 0.52.

Calculate the probability that a policyholder will make a claim on only his homeowner's insurance policy.

## Question 1.9      *SOA/CAS*

The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35%. Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work.

Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

## Question 1.10

A survey of an insurance company's medical policyholders in the last three years revealed the following information:

- 31% made claims for dental care
- 27% made claims for optical care
- 21% made claims for other medical care
- 15% made claims for dental and optical care
- 13% made claims for optical and other medical care
- 11% made claims for dental and other medical care
- 7% made claims for all three sorts of care.

Calculate the percentage of the medical policyholders that made none of the three types of claims during the last year.

## Question 1.11      *SOA/CAS*

A doctor is studying the relationship between blood pressure and heartbeat abnormalities in her patients. She tests a random sample of her patients and notes their blood pressures (high, low or normal) and their heartbeats (regular or irregular). She finds that:

(i) 14% have high blood pressure.

(ii) 22% have low blood pressure.

(iii) 15% have an irregular heartbeat.

(iv) Of those with an irregular heartbeat, one-third have high blood pressure.

(v) Of those with normal blood pressure, one-eighth have an irregular heartbeat.

What portion of the patients selected have a regular heartbeat and low blood pressure?

## Question 1.12

An urn contains 15 balls: 4 red balls numbered 1-4, 5 blue balls numbered 1-5, and 6 yellow balls numbered 1-6.   A ball is selected at random from the urn.

Calculate the probability the random selected ball is yellow or red, and with an even number.

## Question 1.13

An insurance company insures 2,400 policyholders in Ontario of whom 46% are female and 54% are male, and 3,150 policyholders in Quebec, of which 48% are female and 52% are male.

Calculate the probability that a randomly selected policyholder is male.

## Question 1.14

A game is played in which a fair six-sided die is rolled, and a card is selected at random from a standard pack of playing cards.

(i)      Calculate the probability of rolling a 4 and choosing a heart.

(ii)     Calculate the probability of neither rolling a 6 nor choosing a spade.

(iii)    Calculate the probability of rolling no more than a 3 or choosing a red card.

## Question 1.15

The local branch of Fudget Car Rentals owns 20 cars: 10 red, 6 blue, and 4 white.  The local branch of Mavis Car Rentals owns 18 cars: 9 red, 2 blue, and 7 white.  A car rented from Fudget is involved in a road traffic accident with a car rented from Mavis.

Calculate the probability that the two cars involved in the accident are not the same color.

## Question 1.16

Events $A$ and $B$ are independent.  If $\Pr(A) = 0.3$ and $\Pr(B) = 0.4$, calculate $\Pr(A \cap B')$.

## Question 1.17

A fair six-sided die is rolled $n$ times.  Calculate the smallest value of $n$  such that the probability of rolling at least one six is at least 95%.

## Question 1.18

There are two urns.  The first urn contains 6 red balls and 4 blue balls, and the second urn contains 5 red balls and 5 blue balls.  One ball is drawn from the first urn and a second ball is drawn from the second urn.

Calculate the probability that the draw results in at least one blue ball.

## Question 1.19

A printing company has three digital copiers. The probability that any copier will break down on a particular day is 0.05. Assuming that the copiers are independent, calculate the probability that the three copiers all break down on the same day at some point during the next 50 days.

## Question 1.20                                                      *SOA/CAS*

An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn.