Exam P Flashcards

2010 exams

Key concepts

Important formulas

Efficient methods

Advice on exam technique

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HOW TO USE THESE FLASHCARDS

These flashcards are designed to help you to prepare efficiently in the run-up to the Course P exam of the Society of Actuaries. They include conceptual ideas, key formulas and techniques for efficient problem solving. Typical questions on a Course P examination require students to understand and apply several concepts in order to set up a solution, and then perform a series of computations to complete it. So don't look at the lists of formulas as simply being memorization work. There are often simple ideas that underlie the formulas as well as basic mathematical reasons why they are correct. Strive to understand and learn the key relations from this point of view and your knowledge will not be the superficial type that may collapse under the stress of taking the examination. The more that you understand conceptually, the easier it becomes to retain the key ideas and write them down quickly and accurately. Your understanding of probability concepts plays a huge role in Courses M and C that lie ahead where you will encounter more advanced and somewhat abstract probability concepts. You will need a solid foundation to be successful there.

We have designed the flashcards so that they can be carried conveniently and read frequently in the final run-up to the exam, *eg* when sitting on a plane. We hope that you will personalize them by adding your own comments and notes, and checking each section when you feel confident with the material covered.

You will probably also find these summaries useful when you are at the stage of working through the past exams. The BPP Exam P Question and Answer Bank contains a mixture of past exam questions and brand new exam-style questions, along with detailed solutions. By the time you have worked through these questions you will have a clear picture of what the exam is like and what you need to work on to get ready for it. As a final tune-up, try one of the BPP Course P practice exams, containing all new exam-style questions.

Good luck with your studying.

Set theory

The *complement* of subset A is written as A' or $\bar{A}$, and is the subset of all elements that are <u>not</u> elements of A.

The *union* of subsets A and B, which is written as $A \cup B$, is the subset of all elements that are elements of either A or B, or both.

The *intersection* of subsets A and B, which is written as $A \cap B$, is the subset of all elements that are elements of both A and B.

The empty set (a set with no elements) is denoted $\varnothing$.

Sets $A_1, A_2, \cdots$ are *mutually exclusive* if $A_i \cap A_j = \varnothing$ whenever $i \neq j$. The sets are *exhaustive* if their union is S, the entire sample space. If the sets are both mutually exclusive and exhaustive, then they form a *partition* of the sample space.

Basic laws of set theory

1. The *associative laws* state that:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

2. The *distributive laws* state that:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

3. *De Morgan's laws* state that:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

4. If events $A_1, A_2, \cdots$ are mutually exclusive and exhaustive, then:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \cdots$$

Hint: Draw Venn diagrams to help understand these laws.

The multiplication rule

A process involves r steps. If there are n_k ways to complete the k-th step, given that the first $k-1$ steps are completed, then there are $n_1 n_2 \cdots n_r$ ways of completing all steps of the process.

Ordered samples with and without replacement

An *ordered sample* of size k is a selection of k objects from a set of n objects, in which the order of selection *is significant*. A sample is constructed using sampling *with replacement* if any object that is selected is replaced before the next object is selected. A sample is constructed using sampling *without replacement* if any object that is selected is not replaced before the next object is selected.

1. The number of possible ordered samples of size k from a set of n objects, when sampling with replacement is n^k.

2. An ordered sample without replacement is called a *permutation*. The number of possible ordered samples of size k selected from a set of n objects using sampling without replacement is:

$$ {}_nP_k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) $$

Unordered samples without replacement

An *unordered sample* of size k is a selection of k distinct objects from a set of n objects, in which the order of selection is not significant.

3. An unordered sample selected without replacement is called a *combination*. The number of possible unordered samples of size k selected from a set of n objects using sampling without replacement is:

$$ {}_nC_k = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{1\times 2\times\cdots\times k} $$

Another counting rule

4. The number of distinct arrangements of n objects of which n_1 are identical, n_2 are identical, … , and n_r are identical is $\dfrac{n!}{n_1!n_2!\cdots n_r!}$.

Example of probability with equally likely outcomes

A lottery randomly picks a "winning combination" of 6 numbers from the whole numbers 1 through 48. Each possible outcome is equally likely.

The number of outcomes is:

$$\binom{48}{6}=\frac{48\times47\times46\times45\times44\times43}{1\times2\times3\times4\times5\times6}=12{,}271{,}512$$

The probability that a "ticket" (*ie* a combination of 6 particular numbers) is a winning combination is $1/12{,}271{,}512$.

The probability that the winning combination contains exactly 3 single-digit numbers, event E, is computed as follows:

- The number of ways in which exactly 3 of the 9 single-digit numbers can be selected is:

$$\binom{9}{3}=\frac{9\times8\times7}{1\times2\times3}=84$$

- The number of ways in which exactly 3 of the 39 double-digit numbers can be selected is:

$$\binom{39}{3}=\frac{39\times38\times37}{1\times2\times3}=9{,}139$$

So the probability of event E is:

$$\Pr(E)=\frac{84\times9{,}139}{12{,}271{,}512}=0.0626$$

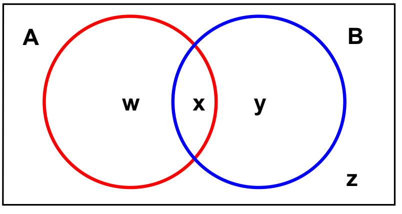
Conditional probability

The *conditional probability of A given B* is denoted $\Pr(A|B)$.

The conditional probability is defined to be:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{where } \Pr(B) > 0$$

If we represent events A and B using a Venn diagram:



then the main relationships are as follows:

$$\Pr(A) = w + x$$

$$\Pr(B) = x + y$$

$$\Pr(A \cap B) = x$$

$$\Pr(A \cup B) = w + x + y = 1 - z$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{x}{x+y}$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{x}{w+x}$$

The *multiplicative law* of probability results from rearranging the conditional probability defintion:

$$\Pr(A \cap B) = \Pr(B)\Pr(A|B) = \Pr(A)\Pr(B|A)$$

Example. Suppose that a purse contains 6 nickels and 10 quarters. What is the probability that the first two coins selected (without replacement) are quarters? Let event A be that the first coin is a quarter, and let event B be that the second coin is a quarter.

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A) = \frac{10}{16} \times \frac{9}{15} = \frac{3}{8}$$

Other properties of conditional probabilities

If events A and B are *independent*, then:

- $\Pr(A|B) = \Pr(A)$
- $\Pr(B|A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A)\Pr(B)$ (multiplicative law)

The basic axioms of probability also apply to general conditional probabilities:

- $\Pr(A|B) \geq 0$
- $\Pr(B|B) = 1$
- If $A_1, A_2, \cdots$ are mutually exclusive events, then:

  $$\Pr(A_1 \cup A_2 \cup \cdots | B) = \Pr(A_1|B) + \Pr(A_2|B) + \cdots$$

- $\Pr(A|B) = 1 - \Pr(A'|B)$
- If the events $A_1, A_2, \cdots, A_n$ are mutually exclusive and exhaustive, then:

  $$\Pr(B) = \sum_{i=1}^{n} \Pr(A_i \cap B) = \sum_{i=1}^{n} \Pr(A_i)\Pr(B|A_i)$$

The normal distribution

The normal distribution is perhaps the most important continuous random variable. The pdf of the normal distribution has two parameters, μ and σ^2, that are the mean and variance.

Probability density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \text{for } -\infty < x < \infty$$

Expectation:

$$E[X] = \mu$$

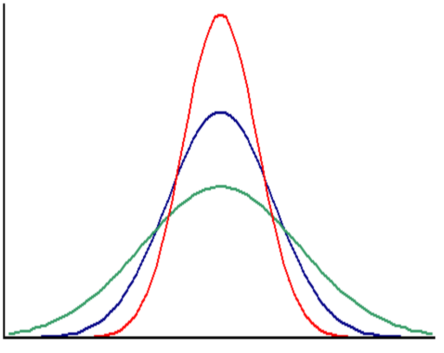
Variance:

$$\operatorname{var}(X) = \sigma^2$$

The notation $X \sim N(\mu, \sigma^2)$ means that the random variable X is normally distributed with mean μ and variance σ^2.

The pdf has a symmetric, bell-shaped graph centered at μ .

A normal distribution with a greater standard deviation has a lower and flatter pdf, because there is a greater chance of observing values far away from the mean. A normal distribution with a lower standard deviation has a higher and narrower pdf, because there is less chance of observing values far away from the mean.



The standard normal distribution

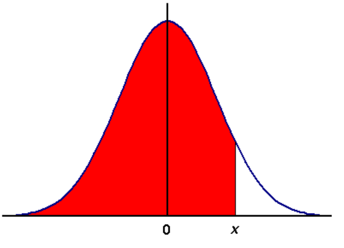
The *standard normal distribution* is a normal distribution with a mean of $\mu = 0$ and a standard deviation of $\sigma = 1$. It has a probability density function, $\phi(x)$, equal to:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right] \qquad \text{for } -\infty < x < \infty$$

The distribution function of the standard normal distribution is usually denoted by $\Phi(x)$:

$$\Phi(x) = \Pr(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds \quad \text{(shaded area below)}$$

This probability cannot be computed using the fundamental theorem of calculus since there is no closed form expression for the anti-derivative.



Numerically approximated values of $\Phi(x)$ for $x = 0, 0.1, 0.2, \cdots, 3.0$ can be found in tables that are distributed with the Course P examination.

To compute values such as $\Phi(-0.3)$ you will need to use the symmetry of the standard normal pdf:

$$\Phi(-x) = \Pr(X \le -x) = \Pr(X \ge x) = 1 - \Phi(x)$$