Solutions to practice questions – Chapter 3

Solution 3.1

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0,1,2 \\ \ddot{a}_{\overline{3}|} & \text{for } K \geq 3 \end{cases}, \text{ where } K = K(30).$$

Solution 3.2

$$E[Y] = \ddot{a}_{\overline{1}|}\, q_{30} + \ddot{a}_{\overline{2}|}\, {}_{1|}q_{30} + \ddot{a}_{\overline{3}|}\, {}_{2}p_{30}$$

$$= 1 + v p_{30} + v^2\, {}_{2}p_{30}$$

Solution 3.3

The current payment method is probably the simplest to use. From the given mortality rates, we have:
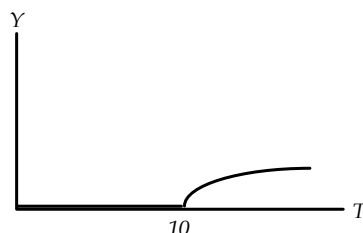
$$p_{30} = 1 - 0.010 = 0.99 \qquad {}_{2}p_{30} = (1 - q_{30})(1 - q_{31}) = 0.99 \times 0.985 = 0.97515$$

So we must have:

$$E[Y] = 1 + v p_{30} + v^2\, {}_{2}p_{30} = 1 + \frac{0.99}{1.05} + \frac{0.97515}{1.05^2} = 2.82735$$

Solution 3.4

$$Y = \begin{cases} 0 & T \leq 10 \\ 1,000\,(\bar{a}_{\overline{T}|} - \bar{a}_{\overline{10}|}) & T > 10 \end{cases} \qquad \text{where } T = T(55)$$

Solution 3.5

$$1000 \,_{10|}\bar{a}_{55} = 1000 \left(\int_0^{10} 0 \, f_{T(55)}(t)\,dt + \int_{10}^{\omega-55} \left(\bar{a}_{\overline{t|}} - \bar{a}_{\overline{10|}} \right) f_{T(55)}(t)\,dt \right)$$

$$= 1000 \int_{10}^{\omega-55} v^t \,_t p_{55}\, dt$$

Solution 3.6

Since $\delta = 0.05$, we have $v^t = e^{-0.05t}$. With de Moivre's law, the future lifetime of (55) is uniformly distributed on the interval $(0, 35]$. The survival function is thus: $_t p_{55} = 1 - t/35$ for $0 < t \le 35$. Evaluating the current payment formula will require an integration by parts to evaluate $\int_{10}^{35} t\, e^{-0.05t}\, dt$. However, evaluating the aggregate payment formula will only require exponential integrals.

$$1000 \,_{10|}\bar{a}_{55} = 1000 \int_{10}^{\omega-55} \left(\bar{a}_{\overline{t|}} - \bar{a}_{\overline{10|}} \right) f_{T(55)}(t)\,dt$$

$$= 1000 \int_{10}^{35} \left(\frac{1 - e^{-0.05t}}{.05} - 7.86939 \right) \frac{1}{35}\, dt$$

$$= \frac{1000}{35} \left(20 \int_{10}^{35} dt - 20 \int_{10}^{35} e^{-0.05t}\, dt - 7.86939 \int_{10}^{35} dt \right)$$

$$= \frac{1000}{35} \left(20 \times 25 + 20 \left(\frac{e^{-0.05t}}{0.05} \bigg|_{10}^{35} \right) - 7.86939 \times 25 \right)$$

$$= 3{,}719$$

Solution 3.7

The fund is $1.1 \times 3{,}719 = 4{,}091$ to the nearest dollar. The life (55) will have to live at least 10 years for Y to exceed this number. So we have:

$$\Pr(Y \le F) = 1 - \Pr(Y > 4{,}091) = 1 - \Pr\left(\bar{a}_{\overline{T|}} - \bar{a}_{\overline{10|}} > 4.091 \right)$$

$$= 1 - \Pr\left(\bar{a}_{\overline{T|}} > 11.960 \right) = 1 - \Pr\left(\frac{1 - e^{-0.05T}}{0.05} > 11.960 \right)$$

$$= 1 - \Pr\left(0.40199 > e^{-0.05T} \right) = 1 - \Pr\left(T > 18.227 \right)$$

$$= 1 - \,_{18.227}p_{55} = 1 - \left(1 - \frac{18.227}{35} \right) = 0.521$$

Solution 3.8

$$\ddot{a}_x = 1 + vp_x + v^2{}_2p_x + \cdots$$
$$= 1 + e^{-\delta}e^{-\mu} + e^{-2\delta}e^{-2\mu} + \cdots \quad \text{(geometric series)}$$
$$= \frac{1}{1 - e^{-(\delta + \mu)}} = \frac{e^{\delta}}{e^{\delta} - e^{-\mu}} = \frac{1+i}{1 + i - (1 - q_x)} = \frac{1+i}{i + q_x}$$

Solution 3.9

Y is a discrete random variable with only 3 possible values:

$Y = 1$ if $K = 0$ and $\Pr(K = 0) = q_{30} = 0.01$

$Y = 1 + v = 1.95238$ if $K = 1$ and $\Pr(K = 1) = {}_{1|}q_{30} = 0.99 \times 0.015 = 0.01485$

$Y = 1 + v + v^2 = 2.85941$ if $K \geq 2$ and $\Pr(K \geq 2) = 0.99 \times 0.985 = 0.97515$

We already have $E[Y] = 2.82735$. The second moment is:

$$E\left[Y^2\right] = 1^2 \times 0.01 + (1.95238)^2 \times 0.01485 + (2.8591)^2 \times 0.97515$$
$$= 8.03965$$

So the variance is $8.03965 - 2.82735^2 = 0.04576$.

Solution 3.10

There is a standard formula for this variance:

$$\text{var}(Y) = 1{,}000^2 \times \frac{{}^2\bar{A}_{55} - \left(\bar{A}_{55}\right)^2}{\delta^2}$$

There is also a shortcut calculation of insurance moments for de Moivre's law (see Section 2.6 of Chapter 2)":

$$\bar{A}_x = \frac{\bar{a}_{\overline{\omega - x|}}}{\omega - x} \quad \Rightarrow$$

$$\bar{A}_{55} = \frac{\bar{a}_{\overline{35|}}}{35} = \frac{1 - e^{-35 \times 0.05}}{35 \times 0.05} = 0.47213$$

$${}^2\bar{A}_{55} = \frac{{}^2\bar{a}_{\overline{35|}}}{35} = \frac{1 - e^{-35 \times 0.10}}{35 \times 0.10} = 0.27709$$

Plug these numbers into the variance formula above and the resulting variance is 21,672,202.

Solution 3.11

The aggregate present value is $S = Y_1 + Y_2 + \cdots + Y_{100}$ where the Y_i are independent. In Solution 3.3 we saw that $E[Y] = 2.82735$. In Solution 3.9 we calculated $\text{var}(Y) = 0.04576$. For the distribution of S, we have:

$$E[S] = 100\,E[Y] = 282.735 \qquad\qquad \text{var}(S) = 100\,\text{var}(Y) = 4.576$$

The fund F is $1.1 \times 282.735 = 311.009$. Since this number is approximately 13 standard deviations above the mean, it is virtually certain that the fund is sufficient.

Solution 3.12

(a) To evaluate this formula, use $v = 1/1.05$, and the UDD rule $_t p_x = 1 - t q_x$ when x is a whole age and t is a fractional part of a year. Then:

$$100\ddot{a}^{(2)}_{30:\overline{3}|} = 50 + 50\,v^{0.5}\,_{0.5}p_{30} + 50\,v\,p_{30} + 50\,v^{1.5}\,_{1.5}p_{30} + 50v^2\,_2p_{30} + 50\,v^{2.5}\,_{2.5}p_{30}$$

$$= 50 + \frac{50 \times (1 - 0.5 \times 0.01)}{1.05^{0.5}} + \frac{50 \times 0.99}{1.05} + \frac{50 \times 0.99 \times (1 - 0.5 \times 0.015)}{1.05^{1.5}}$$

$$+ \frac{50 \times 0.99 \times 0.985}{1.05^2} + \frac{50 \times 0.99 \times 0.985 \times (1 - 0.5 \times 0.020)}{1.05^{2.5}}$$

$$= 278.31$$

(b) Using Woolhouse's approximation:

$$100\ddot{a}^{(2)}_{30:\overline{3}|} = 100\left(\ddot{a}^{(2)}_{30} - v^3\,_3p_{30}\,\ddot{a}^{(2)}_{33}\right)$$

$$\approx 100\left(\ddot{a}_{30} - \frac{1}{4} - v^3\,_3p_{30}\left(\ddot{a}_{33} - \frac{1}{4}\right)\right)$$

$$= 100\left(\ddot{a}_{30:\overline{3}|} - \frac{1}{4}\left(1 - v^3\,_3p_{30}\right)\right)$$

$$= 100\left(1 + \frac{0.99}{1.05} + \frac{0.99 \times 0.985}{1.05^2} - \frac{1}{4}\left(1 - \frac{0.99 \times 0.985 \times 0.98}{1.05^3}\right)\right)$$

$$= 278.37$$

Solution 3.13

Use the insurance shortcut for de Moivre's law:

$$A_{60} = \frac{a\,\overline{_{30|}}}{30} = \frac{1-1.06^{-30}}{30 \times 0.06} = 0.45883 \qquad A_{75} = \frac{a\,\overline{_{15|}}}{15} = \frac{1-1.06^{-15}}{15 \times 0.06} = 0.64748$$

Using the annuity-insurance relation, the value of the whole life annuity due is:

$$1,000\ddot{a}_{60} = 1,000 \times \frac{1-A_{60}}{d} = 9,560.71$$

The value of the certain and life annuity due is:

$$APV = 1,000\left(\ddot{a}\,\overline{_{15|}} + {_{15|}}\ddot{a}_{60}\right) = 1,000\left(\ddot{a}\,\overline{_{15|}} + v^{15}{_{15}}p_{60}\,\ddot{a}_{75}\right)$$

$$= 1,000\left(\frac{1-1.06^{-15}}{0.06/1.06} + 1.06^{-15} \times \frac{15}{30} \times \frac{1-0.64748}{0.06/1.06}\right) = 11,594.30$$

This is approximately a 21.3% increase in the APV.

Solution 3.14

Note first that we have:

$$d^{(2)} = 2\left(1-v^{0.5}\right) = 2\left(1-1.06^{-0.5}\right) = 0.05743$$

Now let's do the APV calculation:

$$APV = 1,000\left(\ddot{a}_{\overline{10|}}^{(2)} + {_{10|}}\ddot{a}_{40}^{(2)}\right) = 1,000\left(\frac{1-v^{10}}{d^{(2)}} + v^{10}{_{10}}p_{40}\,\ddot{a}_{50}^{(2)}\right)$$

$$= 1,000\left(\frac{1-1.06^{-10}}{0.05743} + 0.53667 \times \left(\alpha(2)\ddot{a}_{50} - \beta(2)\right)\right)$$

$$= 1,000\left(7.68968 + 0.53667 \times \left(1.00021 \times 13.2668 - 0.25739\right)\right) = 14,673$$

Solution 3.15

$K(62)$	Probability	Y
0	0.02	50
1	$0.04 - 0.02 = 0.02$	$50 + 75v = 121.42857$
≥ 2	0.96	$50 + 75v + 100v^2 = 212.13152$

Solution 3.16

From the table we have:

$$E[Y]=50\times0.02 + 121.42857\times0.02 + 212.13152\times0.96 = 207.07483$$

$$E\left[Y^2\right]=50^2\times0.02 + 121.42857^2\times0.02 + 212.13152^2\times0.96 = 43,544.69$$

$$\text{var}(Y)=664.70$$

Solution 3.17

With de Moivre's law we have $q_x =1/(\omega-x)=1/(100-x)$. The backward recursion formula is:

$$A_x =vq_x + v\,p_x\,A_{x+1} = \frac{q_x + p_x\,A_{x+1}}{1.05}$$

The starting point is $A_{100}=0$. The recursion formula leads to:

$$A_{99} = \frac{\dfrac{1}{1} + 0\times A_{100}}{1.05} = 0.95238 \qquad A_{98} = \frac{\dfrac{1}{2} + \dfrac{1}{2}\times0.95238}{1.05} = 0.92971$$

$$A_{97} = \frac{\dfrac{1}{3} + \dfrac{2}{3}\times0.92971}{1.05} = 0.90775 \qquad A_{96} = \frac{\dfrac{1}{4} + \dfrac{3}{4}\times0.90775}{1.05} = 0.88649$$

Solution 3.18

You should first notice that p_{81} is also 0.90.

$$e_x =p_x +p_x\,e_{x+1} \;\Rightarrow\; e_{x+1}=\frac{e_x -p_x}{p_x}$$

$$e_{81}=\frac{8.5-0.9}{0.9} = 8.444 \qquad e_{82} =\frac{8.444-0.9}{0.9} = 8.383$$

Solution 3.19

$$F = 1.2 \int_0^5 \underbrace{100(1-0.2w)}_{\text{amount}} \times \underbrace{e^{-0.05w}}_{\text{discount}} \times \underbrace{0.10e^{-0.10w}\,dw}_{\text{probability }f(w)dw}$$

$$= 12\left(\int_0^5 e^{-0.15w}\,dw - 0.2\int_0^5 we^{-0.15w}\,dw\right) = 12\left(\left(-\frac{e^{-0.15w}}{0.15}\bigg|_0^5\right) - 0.2\left(-\frac{e^{-0.05w}(0.15w+1)}{0.15^2}\bigg|_0^5\right)\right)$$

$$= 12(3.51756 - 1.54096) = 23.71910$$

Solution 3.20

(a) $1,000\, A_{45:\overline{5}|}^{1} = 1,000\, v^5\, {}_5p_{45} = \dfrac{1,000\, l_{50}}{1.06^5\, l_{45}} = 729.88$

(b) $\ddot{a}_{45:\overline{5}|} = \ddot{a}_{45} - v^5\, {}_5p_{45}\, \ddot{a}_{50} = 14.1121 - \left(\dfrac{729.88}{1,000}\right)13.2668 = 4.42896$

(c) $\bar{a}_{40:\overline{10}|} = \dfrac{1 - \bar{A}_{40:\overline{10}|}}{\delta} = \dfrac{1 - \left(\dfrac{i}{\delta} A_{40:\overline{10}|}^{1} + A_{40:\overline{10}|}^{1}\right)}{\ln(1.06)}$

$\qquad = \dfrac{1 - \left(\dfrac{0.06}{\ln(1.06)}\left(0.16132 - 0.53667 \times 0.24905\right) + 0.53667\right)}{\ln(1.06)}$

$\qquad = 7.46274$