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Introduction

Before we start the main subject matter in this text, we should take care of a little housekeeping.

Assumed knowledge

We assume that the reader has knowledge of calculus, probability theory, and interest theory. Ideally, your familiarity with these topics should be at a level sufficient for success with Exams P (probability theory) and FM (financial mathematics) of the SOA (or their equivalent). The necessary background information can be found in BPP textbooks written for these two examinations. You can visit our website and download a free sample chapter from these texts or order a copy.

Notation and rounding

We have tried hard to ensure that all new notation is explained clearly. We sometimes use $\exp(x)$ in place of e^x , especially when this avoids complicated superscripts that might otherwise be difficult to read.

Rounding poses a particular dilemma. Our standard policy in this text has been to keep full accuracy within intermediate calculations even though an intermediate result may be shown as a rounded value. So, you may occasionally disagree with the last significant figure or two in a calculation if you calculate the result using the rounded values shown.

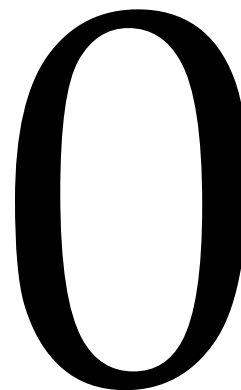
Worked examples and practice questions

Each worked example within the main text is presented in the form of a question that is immediately followed by the solution. You may wish to think about the questions before reading the solutions. The end of each solution is marked with the symbol $\blacklozenge\blacklozenge$ to indicate where the main text continues.

At the end of each chapter we have included some additional practice questions. Short numerical answers to these questions can be found at the end of the book. Detailed worked solutions to these questions can be downloaded free of charge from the BPP Professional Education website at www.bpptraining.com. Other useful study resources can also be found there.

Errors in this text

If you find an error in this text, we'll be pleased to hear from you so that we can publish an errata list for students on our website and correct these errors in the next edition. Please email details of any errors to examMFESupport@bpptraining.com. A current errata list is maintained in the Student Mailbag on the Exam MFE page of the BPP website. Thank you.



Introduction to options

Overview

In this chapter we will study the basic types of options.

By the end of this chapter, you will be able to:

- understand the terminology and notation associated with options
- describe how the basic types of options work
- calculate the payoffs from these options
- identify the cash flows associated with different types of underlying assets
- calculate forward prices and prepaid forward prices.

Rounding convention

In numerical examples, we will carry out the calculations with full accuracy (storing all the intermediate values in calculator memory). However, we cannot show all the decimals in the printed equations. As a result, some of the equations may not appear to “add up” using the values shown. Also, you may get slightly different answers from those shown if you are using a different convention yourself.

0.1 Introduction

Derivatives are a very important class of financial products. These are financial **securities** whose value depends on (is “derived from”) the value of another asset, called the **underlying asset**.

Derivatives include **forwards** and **options**.

In this chapter we will describe the two basic types of options – **call options** and **put options**. We will also explain the terminology relating to options.

We will explain the difference between **European** and **American** options.

Some of the formulas we will be using in later chapters involve **forward prices** and **prepaid forward prices**. So we will describe how to calculate these.

We will conclude with a brief description of the different types of underlying asset on which options may be based.

The standard options described here are usually referred to as **vanilla** options. Options with non-standard features are called **exotic** options. We will study these later in Chapter 6.

0.2 Basic notation

We will use the following notation in relation to options in this course:

- T denotes the time of expiration of the option
- t denotes the current time (so that the remaining life of the option is $T - t$)
- S_t denotes the current market value of the underlying asset
- K denotes the strike price of the option
- r denotes the continuously-compounded risk-free interest rate
- σ denotes the volatility of the underlying asset
- δ denotes the continuously-paid yield on a stock.

In many of our formulas we will assume that $t = 0$, so that the remaining term of the option is simply T .

Unless otherwise stated, we will ignore transaction costs, bid-ask spreads and tax.

0.3 Call and put options

Options allow the owner of the option to buy or sell a specified asset (the underlying asset) at a price that is agreed in advance. If the owner chooses to do this, this is called **exercising** the option.

Options come in two basic types: **call options** and **put options**.

Call and put options

A standard **call option** provides its owner with the right to **purchase** the underlying asset for a fixed price at a specified time, but with no obligation to do so.

A standard **put option** provides its owner with the right to **sell** the underlying asset for a fixed price at a specified time, but with no obligation to do so.

The “specified time” may actually be a range of dates, rather than a single date.

The underlying asset might, for example, be a share of stock in a particular company. We will use S_t to denote the price of the underlying asset at time t .

The agreed price is called the **strike price** or **exercise price**. We will write this as K .

Because owning an option puts the holder in a no-lose situation, a **premium** must usually be paid to obtain the option. The amount paid is called the **option price**. Much of the material in this course is concerned with how we can calculate the fair option price.

Note carefully the distinction between exercising an option and selling an option. If the owner of a call option (say), exercises it, he or she will have to pay the strike price and will receive the underlying asset in return. The option will then cease to exist. If, on the other hand, the owner sells the option, he or she will be transferring it to someone else and will receive a cash payment equal to the prevailing option price. The option itself will continue to exist, but will now have a new owner.

European and American options

Most options have a finite lifetime and the option cannot be exercised once its **expiration date** has passed. We will write the time of expiration as T .

It is important to distinguish between two different “styles” of option, called “European” and “American”.

European and American options

A **European** option can be exercised only on the expiration date, whereas an **American** option can be exercised at any time up to the expiration date.

There is no geographical significance here. These are just conventional names used to distinguish the two different styles.



Example 0.1

Investor A buys 100 call options for \$1 each. Each option allows the holder to purchase 1 share of Company XYZ’s stock in one year’s time at a price of \$5.

Assuming that Investor A still owns the options at the end of the year, determine his overall profit if the stock price at that time is (a) \$7 and (b) \$3.

(In this type of question we ignore interest, which we assume is accounted for separately.)

Solution

In both cases Investor A will have paid \$100 initially.

In scenario (a) the stock price in one year’s time is \$7. Here Investor A will choose to exercise the options, because this will allow him to buy the shares (which are worth \$7 each) at a price of only \$5, resulting in a profit of \$2 per share. His overall profit will then be:

$$\text{Profit} = -\$100 + 100 \times \$2 = +\$100$$

In scenario (b) the stock price in one year’s time is \$3. Here Investor A will choose *not* to exercise the option, because this would mean that he would be paying \$5 for shares that are only worth \$3 each. So he will allow the option to lapse. In this case his overall profit will be:

$$\text{Profit} = -\$100 + 0 = -\$100$$

So, in fact, he has suffered a loss of \$100.

◆◆

Long and short positions

The **holder** of an option is said to have a **long position** in the option. The option holder actually owns the option, which has a positive value in his investment portfolio. Usually the holder will have had to pay a **premium** for the option to obtain it.

However, there must also be a corresponding option **writer**, who has a **short position** in the option. The option writer has **short-sold** the option, which has a negative value in his investment portfolio. The writer will have received the premium paid by the holder.

The activities of the holder and the writer form a zero-sum game, so that the values of their holdings and cash flows always sum to zero, *ie* they always have opposite signs.

Notice that it is always the party with the **long** position who chooses whether to exercise the option. The party with the short position must then comply.

It is also possible to hold a short position in other assets, such as the stock itself. For the purpose of this course you don't need to know how this is actually achieved, just that all the signs are reversed relative to an investor with a long position.



Example 0.2

Investor A (from the previous example) purchased the options from Investor B. Determine the cash flows and the profit or loss Investor B would experience in each of the two scenarios.

Solution

Investor A had a long position in the option, while Investor B had a short position.

In both cases Investor B will have received \$100 initially from Investor A.

In scenario (a) Investor B must hand over to Investor A 100 shares worth \$7 each, but will receive only \$5 per share for them. So Investor B's overall profit will be:

$$\text{Profit} = +\$100 - 100 \times \$2 = -\$100$$

So, in fact, Investor B has suffered a loss of \$100.

In scenario (b) Investor A will choose *not* to exercise the option, so there is no transaction at the expiration date. So Investor B will have made an overall profit of \$100 (from the initial premium). ♦♦

Note that Investor A and Investor B are playing a zero-sum game here. Their combined profit and loss sums to zero. If one investor makes a certain amount of profit, the other makes a loss of the same amount.

Option payoffs

The previous examples illustrate the different possible outcomes for the holder of a European *call* option that expires at time T :

- If $S_T > K$, the option holder can buy the shares at a lower price than they are worth, resulting in a profit on the expiration date of $S_T - K$. In this situation the option is described as being **in-the-money**.
- If $S_T < K$, the option holder should not exercise the option, because this would involve paying more for the shares than they are worth. Here there is no profit or loss at the expiration date. In this situation the option is described as being **out-of-the-money**.

- If $S_T = K$, the option holder should be indifferent to exercising or not exercising the option, as there would be no profit or loss either way. (We are ignoring the effect of transaction costs here.) In this situation the option is described as being **at-the-money**.

This classification into in-/at-/out-of-the-money is referred to as the “moneyness” of the option.

We can summarize these three cases in a single formula:

$$\text{Profit at expiration (time } T \text{) for a call option} = \max(S_T - K, 0)$$

This is the **payoff function** for the option.

Note that the payoff ignores any initial premium paid or received.

Intrinsic value and time value

The value of the payoff function evaluated at any time $t \leq T$, based on S_t , the underlying asset price at that time, is called the **intrinsic value** of the option. So, for a call option, the intrinsic value is:

$$\text{Intrinsic value (at time } t \leq T \text{) of a call option} = \max(S_t - K, 0)$$

Note that we can still calculate the intrinsic value for a European option, even though this type of option cannot actually be exercised before time T .

The difference (at any time $t \leq T$) between the market price of an option and its intrinsic value is called the **time value** of the option. The time value reflects the fact that the intrinsic value could increase further in the future (making the option more valuable), but is underpinned by the value of zero (preventing any further losses beyond this point).

So, for all types of options, we have the following relationship.

Relationship between option price, intrinsic value and time value

$$\text{Option price} = \text{Intrinsic value} + \text{Time value}$$

Here is a summary of the key facts relating to the payoffs of vanilla call and put options.

Vanilla call and put options

	Call option	Put option
Payoff function:	$\max(S_T - K, 0)$	$\max(K - S_T, 0)$
The payoff for an investor with a short position is the negative of the payoff function shown.		
“Moneyness”:		
$S_t > K$	in-the-money	out-of-the-money
$S_t = K$	at-the-money	at-the-money
$S_t < K$	out-of-the-money	in-the-money
Intrinsic value:	$\max(S_t - K, 0)$	$\max(K - S_t, 0)$



Example 0.3

The S&P 500 index is currently standing at 1450. Vanilla put options on this index with a strike price of 1500 are currently trading at a price of 150.

Calculate the intrinsic value and the time value for these options.

Solution

The intrinsic value is:

$$\max(K - S_t, 0) = \max(1500 - 1450, 0) = 50$$

The time value can then be calculated as:

$$\begin{aligned} \text{Time value} &= \text{Option price} - \text{Intrinsic value} \\ &= 150 - 50 = 100 \end{aligned}$$

◆◆

0.4 The underlying asset

Options are available based on a wide range of different types of underlying asset. The ones we will focus on in this course are:

- stocks / shares (or a stock index)
- currencies
- commodities
- bonds.

When we are considering options, an important issue is whether there are any **cash flows** associated with owning the underlying asset. So let's review the situation with each type of asset.

Stocks / shares

With **shares** of company stock, we need to consider whether any **dividends** will be paid during the life of the option. In real life, any dividends must be paid in the form of discrete payments (eg quarterly). However, for the continuous-time models we will study in later chapters, it is often more convenient to treat dividends approximately, by assuming that they are paid in the form of a continuous cash flow. This means that, if a stock with current price S_t has an annual dividend yield of δ , it will pay a dividend equal to $\delta S_t dt$ during each short time period $(t, t + dt)$.

If the underlying asset is a **stock index** calculated as the value of a notional portfolio of stocks (eg the Dow Jones index), δ is the (average) dividend yield for the whole portfolio.

Foreign currencies

The underlying asset could be a unit of a **foreign currency**, eg British pounds. Here we need to take account of the fact that British pounds will earn **interest** (based on the prevailing interest rate in the United Kingdom). In this case it is a very good approximation to assume that interest is earned as a continuous cash flow.

Commodities

Owning a **commodity**, such as gold or coffee beans, does not usually generate any form of income. In fact, the opposite is true. There are **storage costs** and insurance costs, which result in a negative cash flow. Again it is a good approximation to assume that these are incurred continuously.

Bonds

Bonds may be **zero-coupon bonds**, in which case there are no cash flows to worry about, or **coupon bonds**, which will normally generate **interest payments** paid at discrete times (eg quarterly or semi-annually).

0.5 Forward prices

Forward prices

Forward prices

A **forward price** is a price agreed upon now for a transaction scheduled to take place at a specified time in the future.

The forward price is chosen so that there is **no initial cost** in entering into a forward agreement.

Investors can enter into a forward agreement as buyers or sellers of the underlying asset. In either case, there is no initial cost to entering into the agreement. This is in contrast to call and put options, which require a premium to be paid.

Payoffs from a forward contract

If the forward price agreed upon now for the purchase of a share of stock at time T is $F_{0,T}$, the payoffs from the forward agreement at time T for a buyer and a seller are as follows:

$$\text{Payoff to the buyer of the forward} = S_T - F_{0,T}$$

$$\text{Payoff to the seller of the forward} = F_{0,T} - S_T$$

The payoff to the buyer is the opposite of the payoff to the seller.

Notice that the payoff for a forward agreement can be positive or negative, whereas call and put options never have negative payoffs. (We will see one exception to this rule in a later chapter.)

Formulas for the forward price of a stock

There are three formulas for the forward price of a stock, corresponding to three different cases.

Forward price of a stock

The forward price agreed upon now for delivery (and payment) at time T depends on the dividends paid to the stockholders:

- No dividends: $F_{0,T} = S_0 e^{rT}$
- Discrete dividends: $F_{0,T} = S_0 e^{rT} - FV_{0,T}(\text{Div})$
- Continuous dividends: $F_{0,T} = S_0 e^{(r-\delta)T}$

The three cases are as follows:

1. The asset does not pay dividends.

When we say that an asset does not pay dividends, we mean that it does not pay dividends before time T .

2. The asset pays discrete dividends.

If the underlying asset pays n discrete dividends prior to time T , and the amount of the dividend paid at time t_i is D_{t_i} , then the present value of the dividends is:

$$PV_{0,T}(\text{Div}) = \sum_{i=1}^n e^{-rt_i} D_{t_i}$$

The accumulated value (future value) of the dividends is:

$$FV_{0,T}(\text{Div}) = \sum_{i=1}^n e^{r(T-t_i)} D_{t_i}$$

Notice that the present value of the dividends is based on the risk-free rate of return. We are assuming here that the dividends are certain to be paid.

*There is one technicality in relation to discrete dividends that we need to mention. In practice each dividend payment has an associated **ex-dividend date**, which is a few days before the date the dividend will actually be paid. It is the owner of the stock on the ex-dividend date who will receive the dividend. This avoids problems if the stock changes hands just before the dividend is paid. In this course we will ignore this issue and assume that the ex-dividend date and the dividend payment date are the same.*

3. The asset pays continuous dividends.

Continuous dividends are expressed as a percentage of the stock price at the time they are paid.

Note that, in this case, if we started with one share of stock and immediately reinvested the dividends in the stock, at the end of a time period of length T , we would have $e^{\delta T}$ shares, where δ is the annual continuously-compounded dividend rate. Therefore, if we want to have 1 share at time T , we would need to buy only $e^{-\delta T}$ shares at time 0.



Example 0.4

A stock currently has a market price of \$25 and will pay a dividend of \$1 in 4 months' time. The continuously-compounded risk-free rate of interest will have a constant value of 6% per annum over the next year.

Calculate the 3-month and 6-month forward prices for this stock.

Solution

There are no dividends paid during the next 3 months. So the 3-month forward price can be found using the "No dividends" formula:

$$F_{0,T} = S_0 e^{rT}$$

So: $F_{0,0.25} = 25e^{0.06 \times 3/12} = \25.38

The dividend of \$1 is a "lump sum" payment that falls within the 6-month window. So, for the 6-month forward price, we need to use the "Discrete dividends" formula:

$$F_{0,T} = S_0 e^{rT} - FV_{0,T}(\text{Div})$$

So: $F_{0,0.5} = 25e^{0.06 \times 6/12} - 1e^{0.06 \times 2/12} = 25.76 - 1.01 = \24.75 ◆◆



Example 0.5

Explain how to derive the formula $F_{0,T} = S_0 e^{(r-\delta)T}$, which gives the forward price of a stock that pays dividends continuously.

Solution

Suppose that Investor 1 enters into a long forward agreement (which requires no initial payment) at time 0 when the forward price is $F_{0,T}$ and at the same time deposits $F_{0,T} e^{-rT}$ in cash at the bank. So this will cost him $F_{0,T} e^{-rT}$ initially. At time T , the cash will have accumulated to $F_{0,T}$, which is the amount he will need to honor the forward agreement and convert it into 1 share of the stock.

Suppose that Investor 2 buys $e^{-\delta T}$ shares at time 0 and reinvests the dividends as they are received. So this will cost him $S_0 e^{-\delta T}$ initially. At time T , Investor 2 will also have 1 share of the stock. (See the note in 3. above.)

Since the two investors end up in exactly the same position, we would expect their initial set-up costs to match up as well, ie $F_{0,T} e^{-rT} = S_0 e^{-\delta T}$. Rearranging this equation then gives the required formula for the forward price:

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

The argument we have used here relies on the concept of “no arbitrage”, which we will look at in more detail in the next chapter. ♦♦

Forward prices for other underlying assets

For other underlying assets, we can calculate the forward price in a similar way, making an adjustment for any cash flows associated with owning the asset, as we did for the dividends on stocks.

Prepaid forward prices

A **prepaid forward price** is the amount that should be paid *now* in order to receive an asset at a specified time in the future.

A prepaid forward price differs from a forward price because a forward price is paid when the asset is delivered, whereas a prepaid forward price is paid prior to the delivery of the asset. The prepaid forward price is just the present value of the forward price.

In real life, payments for forward contracts are made at the time of delivery. However, we will see later that some of the formulas for options are understood most easily if we think in terms of prepaid forward prices.

Prepaid forward price

The prepaid forward price, $F_{0,T}^P$, paid at time 0 for an asset to be delivered at time T depends on whether the asset pays dividends and whether those dividends are discrete or continuous:

- No dividends: $F_{0,T}^P = S_0$
- Discrete dividends: $F_{0,T}^P = S_0 - PV_{0,T}(Div)$
- Continuous dividends: $F_{0,T}^P = S_0 e^{-\delta T}$

The “P” superscript indicates a prepaid forward price.

**Example 0.6**

A stock currently has a market price of \$10. The continuously-compounded risk-free rate of interest will have a constant value of 6% per annum over the next year.

Calculate the one-year prepaid forward price for the stock, if it is assumed that the stock pays dividends continuously at a rate of 4% per annum.

Solution

Here we need to use the “Continuous dividends” formula, which gives:

$$F_{0,T}^P = S_0 e^{-\delta T}$$

So: $F_{0,1}^P = 10e^{-0.04 \times 1} = \9.61 ◆◆

Note that, in this calculation, we didn’t actually need to know the risk-free interest rate.

0.6 Uses of options

Reasons for using options

The main reasons for using options are:

- hedging (to reduce risk)
- speculation (to bet on a market movement)
- arbitrage (to make a risk-free profit)
- portfolio management (to adjust the exposures to different factors in a portfolio).

We will have more to say about these later.

Chapter 0 Practice Questions

Question 0.1

Fill in the gaps in the paragraph below:

A call option gives the holder the right to _____ the underlying asset at an agreed price. This agreed price is called the _____. The option will be out-of-the-money if the underlying asset price is _____. Exercise is permitted only on the expiration date, so the style of this option is _____. The payoff from this option for an investor who has a short position in 10 options is equal to _____.

Question 0.2

Investor C buys 100 put options with a strike price of \$5 for \$1 each. Determine the highest overall profit and the highest overall loss Investor C could make on this investment.

Question 0.3

Investor D sells 100 call options with a strike price of \$5 for \$1 each. Determine the highest overall profit and the highest overall loss Investor D could make on this investment.

Question 0.4

What relationship would you expect to apply between the prices of otherwise identical European and American options?

Question 0.5

Investor E has a long position in 2000 call options with a strike price of \$10 and a short position in 1000 put options with a strike price of \$5. Both options are on the same underlying asset.

- (i) Write down a formula for the payoff function for Investor E in terms of the underlying asset price S .
- (ii) Sketch a graph of the payoff function (as a function of the price S).

Question 0.6

What combination of call and/or put options would have a payoff function of $|S_T - K|$?

Question 0.7

Investor F owns:

- 500 put options with a strike price of \$600, which he purchased for \$25 each
- 250 put options with a strike price of \$700, which he purchased for \$100 each.

The underlying asset for each option is 1 ounce of gold, whose current price is \$675. Determine the intrinsic value for Investor F's portfolio of gold options.

Question 0.8

What additional information would you need in order to find the time value of Investor F's portfolio?

Question 0.9

What can you say about the time value of an American option

- (a) at its expiration date
- (b) before its expiration date?

Question 0.10

Calculate the 3-month forward price for each of the following assets. Where relevant, assume that the risk-free continuously-compounded rate of interest has a constant value of 4% per annum in the United States and 6% in the United Kingdom.

- (a) 1 million shares of Google Inc's stock, which currently has a market price of \$475 per share, and is not expected to pay a dividend in the next 3 months
- (b) 1 ton of coffee beans, which currently have a market price of \$1500, and would cost \$25 (payable in advance) to store for the next 3 months
- (c) 1 million GBP (British pounds), which are currently equivalent to \$1.95 million.

Question 0.11

Calculate the prepaid forward price for each of the assets in the previous question.

Question 0.12

Suggest a reason why an individual or an organization might wish to undertake each of the following transactions:

- (a) purchase a put option on the NASDAQ index
- (b) purchase a call option on 1000 Microsoft shares
- (c) sell put options on 100 ounces of platinum.