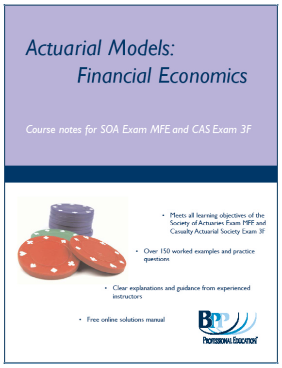

Financial economics (MFE)

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Solutions to practice questions – Chapter 7

Solution 7.1

$Z(1)^2$ will be greater than 4 if $Z(1)$ is greater than 2 or if it is less than –2:

$$P[Z(1)^2 > 4] = P[Z(1) > 2] + P[Z(1) < -2]$$

Since $Z(0) = 0$, $Z(1)$ is the same as the increment $Z(1) - Z(0)$, and this has a $N(0,1)$ distribution.

So: $P[Z(1) > 2] = P[N(0,1) > 2] = 0.0228$ (from statistical tables)

and $P[Z(1) < -2] = P[N(0,1) < -2] = 0.0228$

So: $P[Z(1)^2 > 4] = 0.0228 + 0.0228 = 0.0456$

Solution 7.2

We can use the formula $\rho = \min\left(\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_1}}\right)$, with $t_1 = 100$ and $t_2 = 101$. This gives:

$$\rho = \min\left(\sqrt{\frac{100}{101}}, \sqrt{\frac{101}{100}}\right) = \sqrt{\frac{100}{101}} = 0.995$$

So the values of $Z(100)$ and $Z(101)$ are almost perfectly correlated.

Solution 7.3

Process (b) mean-reverts to a value of 1.

If $X(t) < 1$, the drift $0.1[1 - X(t)]$ has a positive value, pushing the process upwards (towards 1).

If $X(t) > 1$, the drift $0.1[1 - X(t)]$ has a negative value, pushing the process downwards (again towards 1).

The drift in process (a) always directs the process values *away* from the value 1. Process (c) and process (d) have a constant drift, and so cannot be mean-reverting.

Solution 7.4

Process (a) can be written in the form $dX(t) = 0.05X(t)dt + 0.05X(t)dZ(t)$. So its drift and volatility are both equal to $0.05X(t)$. Since these are both proportional to the process value $X(t)$, this is geometric Brownian motion.

Process (b) can be written in the form $dX(t) = -0.05X(t)dt + 0.05dZ(t)$. Its drift is $-0.05X(t)$, which is of the form $\lambda[\alpha - X(t)]$, with $\alpha = 0$ and $\lambda = 0.05$, and its volatility is constant. So this is an Ornstein-Uhlenbeck process.

Process (c) can be written in the form $dX(t) = 0.05dt + 0.05dZ(t)$. Its drift and volatility both have constant values of 0.05. So this is arithmetic Brownian motion.

Solution 7.5

If $C[S(t),t] = t\,S(t)^2$, then:

$$C_S = \frac{\partial C}{\partial S} = 2tS$$

$$C_{SS} = \frac{\partial^2 C}{\partial S^2} = 2t$$

and $$C_t = \frac{\partial C}{\partial t} = S^2$$

So the SDE for $C[S(t),t] = t\,S(t)^2$ is:

$$dC(S,t) = C_S dS + \tfrac{1}{2} C_{SS}(dS)^2 + C_t dt$$

$$d\left[t\,S(t)^2\right] = 2tSdS + \tfrac{1}{2}(2t)(dS)^2 + S^2 dt$$

$$= 2tSdS + t(dS)^2 + S^2 dt$$

We can now use the SDE given for the process $S(t)$ to simplify this further:

$$d\left[t\,S(t)^2\right] = 2tS[5dt + 10dZ(t)] + t[5dt + 10dZ(t)]^2 + S^2 dt$$

$$= 10tS\,dt + 20tSdZ(t) + t(100dt) + S^2 dt$$

$$= \left[100t + 10tS + S^2\right]dt + 20tSdZ(t)$$

To simplify the squared term in the first line, we've used the relationships $(dt)^2 = dt\,dZ(t) = 0$ *and* $[dZ(t)]^2 = dt$.

Solution 7.6

If $C[Z(t),t]=e^{2Z(t)}$, then:

$$C_Z=\frac{\partial C}{\partial Z}=2e^{2Z(t)}$$

$$C_{ZZ}=\frac{\partial^2 C}{\partial Z^2}=4e^{2Z(t)}$$

and $$C_t=\frac{\partial C}{\partial t}=0$$

So the SDE for $C[Z(t),t]=e^{2Z(t)}$ is:

$$dC(Z,t)=C_Z dZ+\tfrac{1}{2}C_{ZZ}(dZ)^2+C_t dt$$

$$\begin{aligned} d\left[e^{2Z(t)}\right] &= 2e^{2Z(t)}dZ+\tfrac{1}{2}(4e^{2Z(t)})(dZ)^2+0dt \\ &= 2e^{2Z(t)}dZ+2e^{2Z(t)}dt \\ &= 2e^{2Z(t)}dt+2e^{2Z(t)}dZ(t) \end{aligned}$$

Again, we've used the relationship $[dZ(t)]^2=dt$.

Solution 7.7

If $C[Z(t),t]=\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]$, then:

$$C_Z=\frac{\partial C}{\partial Z}=\lambda\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]=\lambda C$$

$$C_{ZZ}=\frac{\partial^2 C}{\partial Z^2}=\lambda^2\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]=\lambda^2 C$$

and $$C_t=\frac{\partial C}{\partial t}=-0.5\lambda^2\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]=-0.5\lambda^2 C$$

So the SDE for $C[Z(t),t]=\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]$ is:

$$dC(Z,t)=C_Z dZ+\tfrac{1}{2}C_{ZZ}(dZ)^2+C_t dt$$

$$\begin{aligned} d\left[\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]\right] &= \lambda C dZ+\cancel{\tfrac{1}{2}\lambda^2 C\underbrace{(dZ)^2}_{=dt}}-\cancel{0.5\lambda^2 C dt} \\ &= 0dt+\lambda C[Z(t),t]dZ(t) \quad \text{or} \quad 0dt+\lambda\exp\left[\lambda Z(t)-0.5\lambda^2 t\right]dZ(t) \end{aligned}$$

The coefficient of dt *in this equation is 0. So the process has no drift, ie it is a martingale.*

Solution 7.8

The Sharpe ratio is defined as:

$$\textit{Sharpe ratio} = \frac{\alpha - r}{\sigma}$$

where α is the continuously-compounded expected rate of return for the asset

σ is the volatility of the asset

and r is the continuously-compounded risk-free interest rate.

The key property of the Sharpe ratio is that the prices of assets that are perfectly correlated must have the same Sharpe ratio.

If the underlying asset price goes up, the call option price will go up and the put option price will go down. The prices of the two options are perfectly (negatively) correlated. So they will have the same Sharpe ratio.

Note that it doesn't matter whether the movements are positively or negatively correlated. So long as they are perfectly correlated, the Sharpe ratio will be the same.

Solution 7.9

We can calculate the price of this option using the power call option formula:

$$C_{Eur}^{power} = S^n e^{-\delta_n T} N(d_1) - Ke^{-rT} N(d_2)$$

The parameter values here are:

$n = 0.5$, $S(0) = 100$, $K = 10$, $T = 0.5$, $r = 0.05$, $\delta = 0$ and $\sigma = 0.4$.

We first need to find:

$$\sigma_n = n\sigma, \ \textit{ie} \ \ \sigma_{0.5} = 0.5\sigma = 0.5(0.4) = 0.2,$$

and $$\delta_n = n\delta - (n-1)\left(r + \tfrac{1}{2}n\sigma^2\right), \ \textit{ie} \ \ \delta_{0.5} = 0.5(0) - (0.5-1)\left(0.05 + \tfrac{1}{2}(0.5)(0.4)^2\right) = 0.045$$

So: $$d_1 = \frac{\ln\left(\dfrac{S^n}{K}\right) + \left(r - \delta_n + \frac{1}{2}\sigma_n^2\right)T}{\sigma_n\sqrt{T}} = \frac{\ln\left(\dfrac{100^{0.5}}{10}\right) + \left(0.05 - 0.045 + \frac{1}{2}\times 0.2^2\right)(0.5)}{0.2\sqrt{0.5}} = 0.09$$

and $$d_2 = d_1 - \sigma_n\sqrt{T} = 0.09 - 0.2\sqrt{0.5} = -0.05$$

So: $$\begin{aligned} C_{Eur}^{power} &= S^n e^{-\delta_n T} N(d_1) - Ke^{-rT} N(d_2) \\ &= 100^{0.5} e^{-0.045(0.5)} N(0.09) - 10e^{-0.05(0.5)} N(-0.05) \\ &= 9.78 \times 0.5359 - 9.75 \times 0.4801 \\ &= 0.56 \end{aligned}$$

Solution 7.10

(a) Under the risk-neutral probability measure, asset prices earn the risk-free rate on average. So we just need to replace the 0.05 coefficient in the real-world SDE with the risk-free rate 0.04, to get:

$$\frac{dS(t)}{S(t)} = 0.04dt + 0.2dZ(t)^*$$

Alternatively, we can see from the SDE given that the Sharpe ratio is $\frac{0.05-0.04}{0.2} = 0.05$.

So Girsanov's theorem tells us that the real-world and risk-neutral Brownian increments are related by the equation $dZ(t)^* = dZ(t) + 0.05dt$. Rearranging this, and substituting into the equation given, we get:

$$\frac{dS(t)}{S(t)} = 0.05dt + 0.2dZ(t) = 0.05dt + 0.2\left[dZ(t)^* - 0.05dt\right] = 0.04dt + 0.2dZ(t)^*$$

(b) The solution to the general SDE for geometric Brownian motion, $\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$, is $S(t) = S(0)\exp\left[\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma Z(t)\right]$. So here we have:

$$S(t) = 5\exp\left[\left(0.05 - \tfrac{1}{2}\times 0.2^2\right)t + 0.2Z(t)\right] = 5\exp\left[0.03t + 0.2Z(t)\right]$$

(c) Taking logs of the solution in part (b) gives:

$$\ln S(t) = \ln 5 + 0.03t + 0.2Z(t)$$

Under the real-world probability measure, $Z(t) \sim N(0,t)$, and so the variance of this is:

$$\text{var}\left[\ln S(t)\right] = \text{var}\left[\ln 5 + 0.03t + 0.2Z(t)\right] = 0.2^2 \ \text{var}\left[Z(t)\right] = 0.04t$$

Note that, if we do this calculation under the risk-neutral probability measure, we get the same answer. The variance of $\ln S(t)$ *is the same under the real-world and risk-neutral measures. Only the drift (expected rate of return) changes.*

(d) Using the solution in part (b):

$$E\left[S(t)^2\right] = E\left[25e^{0.06t+0.4Z(t)}\right] = 25e^{0.06t}E\left[e^{0.4Z(t)}\right]$$

To simplify this, we can use the result that, if $X \sim N(\mu,\sigma^2)$, then $E\left[e^{kX}\right] = \exp\left[k\mu + \frac{1}{2}k^2\sigma^2\right]$. So this gives:

$$E\left[S(t)^2\right] = 25e^{0.06t}\exp\left[\tfrac{1}{2}\times 0.4^2 t\right] = 25e^{0.14t}$$

(e) The corresponding solution to the risk-neutral SDE is:

$$S(t) = 5\exp\left[0.02t + 0.2Z(t)^*\right]$$

So, using the same method:

$$E^*\left[\sqrt{S(t)}\right] = E^*\left[S(t)^{0.5}\right] = E\left[\sqrt{5}\,e^{0.01t+0.1Z(t)^*}\right] = \sqrt{5}\,e^{0.01t}E\left[e^{0.1Z(t)^*}\right] = \sqrt{5}\,e^{0.01t}e^{\frac{1}{2}\times 0.1^2 t} = \sqrt{5}\,e^{0.015t}$$

Solution 7.11

The quadratic variation of $S(t)$ over the time period $(0,1)$ is:

$$\begin{aligned}\int_0^1 [dS(t)]^2 &= \int_0^1 S(t)^2 \,[0.05dt + 0.2dZ(t)]^2 \\ &= \int_0^1 S(t)^2 \times 0.2^2 \, dt \\ &= 0.04\int_0^1 S(t)^2 \, dt\end{aligned}$$

The solution to the SDE given is:

$$S(t) = S(0)\exp\left[\left(0.05 - \tfrac{1}{2}\times 0.2^2\right)t + 0.2Z(t)\right] = 5e^{0.03t+0.2Z(t)}$$

So the quadratic variation is:

$$\begin{aligned}0.04\int_0^1 S(t)^2 \, dt &= 0.04\int_0^1 \left[5e^{0.03t+0.2Z(t)}\right]^2 \, dt \\ &= 0.04\times 5^2 \int_0^1 e^{0.06t+0.4Z(t)} dt \\ &= \int_0^1 e^{0.06t+0.4Z(t)} dt\end{aligned}$$

Since $Z(t)$, which has a $N(0,t)$ distribution, behaves randomly, the quadratic variation is a random variable. Its expected value is:

$$E\left[\int_0^1 e^{0.06t+0.4Z(t)} dt\right] = \int_0^1 E\left[e^{0.06t+0.4Z(t)}\right] dt = \int_0^1 e^{0.06t} E\left[e^{0.4Z(t)}\right] dt$$

If $X \sim N(\mu,\sigma^2)$, then $E\left[e^{kX}\right] = e^{k\mu+\frac{1}{2}k^2\sigma^2}$. So the expected value of the quadratic variation is:

$$\int_0^1 e^{0.06t} e^{\frac{1}{2}\times 0.4^2 t} dt = \int_0^1 e^{0.14t} dt = \left.\frac{e^{0.14t}}{0.14}\right|_0^1 = \frac{e^{0.14} - 1}{0.14} = 1.07$$
