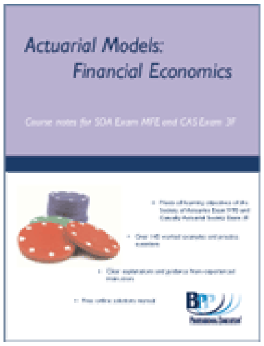


Actuarial Mathematics: Financial Economics (MFE)

Published by BPP Professional Education

Solutions to practice questions – Chapter 3

Solution 3.1

With a **one-step binomial tree**, the results of the calculations are as follows:

Based on these parameter values, we have:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0)(1)+0.2\sqrt{1}} = 1.2840$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.05-0)(1)-0.2\sqrt{1}} = 0.8607$$

We can then calculate the future prices of the underlying asset and the payoffs at the end of one year. (Remember that it's an at-the-money option, so $K = S_0$.)

The risk-neutral probabilities are:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.05-0)(1)} - 0.8607}{1.2840 - 0.8607} = 0.4502 \quad \text{and} \quad 1 - p^* = 0.5498$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(1)}\left[0.4502 \times 0 + 0.5498 \times 13.93\right] = 7.29$$

Solution 3.2

With a **two-step binomial tree**, the results of the calculations are as follows:

Now	*6 months*	*1 year*
		0.00
	0.00	139.49
5.66	118.11	*0.00*
100.00	*10.84*	105.13
	89.01	*20.77*
		79.23

Here, we have $h = 0.5$, so that:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0)(0.5)+0.2\sqrt{0.5}} = 1.1811$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.05-0)(0.5)-0.2\sqrt{0.5}} = 0.8901$$

We can then calculate the future prices of the underlying asset and the payoffs at the end of one year. For example, at the 79.23 node:

$$S_0 dd = 100 \times 0.8901^2 = 79.23 \quad \text{and} \quad \max(K - S_T, 0) = \max(100 - 79.23, 0) = 20.77$$

The risk-neutral probabilities are:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0)(0.5)} - 0.8901}{1.1811 - 0.8901} = 0.4647 \quad \text{and} \quad 1 - p^* = 0.5353$$

The option prices at the earlier nodes are then calculated by discounting the expected values stemming from that node. For example, at the 89.01 node:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4647 \times 0 + 0.5353 \times 20.77\right] = 10.84$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4647 \times 0 + 0.5353 \times 10.84\right] = 5.66$$

Since there is only one node in this example with a nonzero payoff at the expiration date, we could have calculated the initial value of the option more quickly here. The node where the payoff is 20.77 is reached if the path is "down-down", which has a risk-neutral probability of $(1-p^*)^2$. *So the discounted payoff from the option (discounted for a full year) is:*

$$C = e^{-2rh}\left[(1-p^*)^2 C_{dd}\right] = e^{-2(0.05)(0.5)}\left[0.5353^2 \times 20.77\right] = 5.66 \quad (\textit{as before})$$

Solution 3.3

With a **three-step binomial tree**, the results of the calculations are as follows:

Now	*4 months*	*8 months*	*1 year*
			0.00
		0.00	148.65
	1.71	130.25	*0.00*
5.99	114.13	3.30	117.99
100.00	*10.00*	103.39	*6.34*
	90.59	*16.28*	93.66
		82.07	*25.65*
			74.35

Here, we have $h = \frac{1}{3}$, so that:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{(0.05-0)\left(\frac{1}{3}\right)+0.2\sqrt{\frac{1}{3}}} = 1.1413$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{(0.05-0)\left(\frac{1}{3}\right)-0.2\sqrt{\frac{1}{3}}} = 0.9059$$

We can then calculate the future prices of the underlying asset and the payoffs at the end of one year. For example, at the 93.66 node:

$$S_0udd = 100 \times 1.1413 \times 0.8901^2 = 93.66 \quad \text{and} \quad \max(K - S_T, 0) = \max(100 - 93.66, 0) = 6.34$$

The risk-neutral probabilities are:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0)\left(\frac{1}{3}\right)} - 0.9059}{1.1413 - 0.9059} = 0.4712 \quad \text{and} \quad 1 - p^* = 0.5288$$

The option prices at the earlier nodes are then calculated by discounting the expected values stemming from that node. For example, at the 82.07 node:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05\left(\frac{1}{3}\right)}\left[0.4712 \times 6.34 + 0.5288 \times 25.65\right] = 16.28$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05\left(\frac{1}{3}\right)}\left[0.4712 \times 1.71 + 0.5288 \times 10.00\right] = 5.99$$

If you compare the answers we obtained using a one-step, a two-step and a three-step tree (which were 7.29, 5.66 and 5.99), you will see that they show an oscillating pattern. The answers alternately increase and decrease as we add extra steps. This is a common feature of binomial trees.

Solution 3.4

(a) With the **Cox-Ross-Rubinstein model**, the results of the calculations are as follows:

Now	*1 year*
	0.00
7.29	122.14
100.00	*18.13*
	81.87

Here, we have:

$$u = e^{\sigma\sqrt{h}} = e^{0.2\sqrt{1}} = 1.2214$$
$$d = e^{-\sigma\sqrt{h}} = e^{-0.2\sqrt{1}} = 0.8187$$

The risk-neutral probabilities are:

$$p^* = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05(1)} - 0.8187}{1.2214 - 0.8187} = 0.5775 \quad \text{and} \quad 1 - p^* = 0.4225$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(1)}\left[0.5775 \times 0 + 0.4225 \times 18.13\right] = 7.29$$

In this case, we get the same answer as with the standard method used in Solution 3.1.

(b) With the **lognormal binomial tree model**, the results of the calculations are as follows:

Now	*1 year*
	0.00
7.43	125.86
100.00	*15.63*
	84.37

Here, we have:

$$u = e^{(r-\delta-½\sigma^2)h+\sigma\sqrt{h}} = e^{(0.05-0-½\times 0.2^2)(1)+0.2\sqrt{1}} = 1.2586$$
$$d = e^{(r-\delta-½\sigma^2)h-\sigma\sqrt{h}} = e^{(0.05-0-½\times 0.2^2)(1)-0.2\sqrt{1}} = 0.8437$$

The risk neutral-probabilities are:

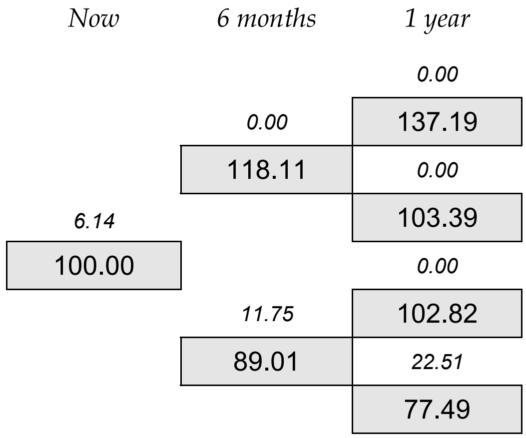
$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.05-0)(1)} - 0.8437}{1.2586 - 0.8437} = 0.5003 \quad \text{and} \quad 1 - p^* = 0.4997$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(1)}\left[0.5003 \times 0 + 0.4997 \times 15.63\right] = 7.43$$

Solution 3.5

(a) Using the **non-recombining method**, the results of the calculations are as follows:



With this method, we only need to adjust the share prices in the tree at the final node, because this is the period in which the dividend is paid.

The 137.19 and 103.39 figures are calculated as:

$$S_{t+h}^{u} = \left[S_t e^{rh} - FV_{t,t+h}(Div)\right] e^{\sigma\sqrt{h}} = \left[118.11 e^{0.05(0.5)} - 2\right] e^{0.2\sqrt{0.5}} = 137.19$$

$$S_{t+h}^{d} = \left[S_t e^{rh} - FV_{t,t+h}(Div)\right] e^{-\sigma\sqrt{h}} = \left[118.11 e^{0.05(0.5)} - 2\right] e^{-0.2\sqrt{0.5}} = 103.39$$

You can see that the figures of 103.39 and 102.82 in this tree are not equal. So the tree doesn't recombine.

We can then fill in the payoffs at the end of one year. For example, at the 77.49 node:

$$\max(K - S_T, 0) = \max(100 - 77.49, 0) = 22.51$$

The risk-neutral probabilities are:

$$p^* = \frac{e^{rh} - d}{u - d} \text{ or } \frac{1 - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}} = 0.4647 \quad \text{and} \quad 1 - p^* = 0.5353$$

The values of p^* and $1-p^*$ are the same as we calculated earlier in Solution 3.2.

The option prices at the earlier nodes are then calculated by discounting the expected values stemming from that node. For example, at the 89.01 node:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4647 \times 0 + 0.5363 \times 22.51\right] = 11.75$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4647 \times 0 + 0.5353 \times 11.75\right] = 6.14$$

(b) Using the **recombining method**, the results of the calculations are as follows:

Now	*6 months*	*1 year*
		0.00
	0.00	137.59
5.66	116.18	*0.00*
98.10	*11.87*	103.13
	87.08	*22.70*
		77.30

With this method, the tree shows prepaid forward prices, so all the entries need to be recalculated.

We have:

$$F_{0,T}^P = S_0 - PV_{0,T}(Div) = 100 - 2e^{-0.05} = 98.10$$

$$\sigma_F = \sigma_S \times \frac{S_0}{F_{0,T}^P} = 0.2 \times \frac{100}{98.10} = 0.2039$$

$$u = e^{rh+\sigma_F\sqrt{h}} = e^{0.05(0.5)+0.2039\sqrt{0.5}} = 1.1843$$
$$d = e^{rh-\sigma_F\sqrt{h}} = e^{0.05(0.5)-0.2039\sqrt{0.5}} = 0.8877$$

We can then fill in the future values in the tree of the prepaid forward price using the u and d factors and the payoffs at the end of one year. For example, at the 77.30 node:

$$98.10 \times d^2 = 98.10 \times 0.8870^2 = 77.30 \quad \text{and} \quad \max(K - S_T, 0) = \max(100 - 77.30, 0) = 22.70$$

The risk-neutral probabilities are:

$$p^* = \frac{e^{rh} - d}{u - d} \text{ or } \frac{1 - e^{-\sigma_F\sqrt{h}}}{e^{\sigma_F\sqrt{h}} - e^{-\sigma_F\sqrt{h}}} = 0.4640 \quad \text{and} \quad 1 - p^* = 0.5360$$

The option prices at the earlier nodes are then calculated by discounting the expected values stemming from that node. For example, at the 87.08 node:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4640 \times 0 + 0.5360 \times 22.70\right] = 11.87$$

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05(0.5)}\left[0.4640 \times 0 + 0.5360 \times 11.87\right] = 6.20$$

This gives a slightly different answer from the one obtained in part (a) because the volatility adjustment is not perfect.

Solution 3.6

With a three-step binomial tree for an **American option**, the results of the calculations are as follows:

Now	*4 months*	*8 months*	*1 year*
			0.00
		0.00	148.65
	1.71	130.25	0.00
~~5.99~~ 6.44	114.13	3.30	117.99
100.00	~~10.00~~ 10.85	103.39	6.34
	90.59	~~16.28~~ 17.93	93.66
		82.07	25.65
			74.35

The basic calculations are the same as for the European option.

However, at the 82.07 node, the exercise value is:

$$\max(K - S_{2h}, 0) = \max(100 - 82.07, 0) = 17.93 \ (> 16.28)$$

Since the exercise value is greater than the value we calculated previously (which assumes that we continue to hold the option), we need to replace the 16.28 with 17.93. This new value is then used in subsequent calculations. So, at the 90.59 node, we have:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05\left(\frac{1}{3}\right)}\left[0.4712 \times 3.30 + 0.5288 \times \underline{17.93}\right] = 10.85 \ (> 9.41)$$

At this node, the exercise value is lower, so we stick with the 10.85 figure.

The initial option price is:

$$C = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right] = e^{-0.05\left(\frac{1}{3}\right)}\left[0.4712 \times 1.71 + 0.5288 \times 10.85\right] = 6.44 \ (> 0)$$

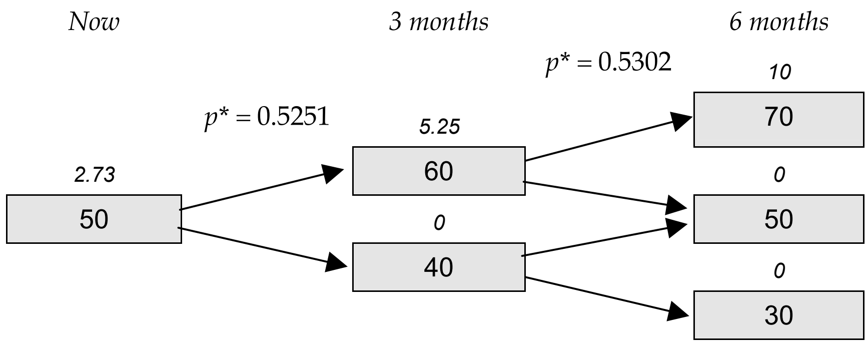
Solution 3.7

(a) In a recombining tree, we gain one extra node as we move from one column to the next. So a 50-step tree will have 51 nodes in the final column (which is quite manageable).

(b) In a *non*-recombining tree, the number of nodes doubles as we move from one column to the next. So a 50-step tree will have 2^{50} nodes in the final column. This number is approximately equal to 10^{15}! So this model could not be used in practice.

Solution 3.8

In this model the values of u and d are not constant throughout the tree. For example, for the up movement from the 50 to the 60 node, the price ratio is $u = \frac{60}{50} = 1.2$, whereas for the up movement from the 60 to the 70 node, it is $u = \frac{70}{60} = 1.167$. As a result, the risk-neutral probabilities will be different for each of the sub-branches in the tree.



For the 50-60-40 branch at the start of the tree, we have:

$$u = \frac{60}{50} = 1.2\,,\quad d = \frac{40}{50} = 0.8$$

So the risk-neutral probabilities are:

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{(0.04-0)(0.25)} - 0.8}{1.2-0.8} = 0.5251 \quad \text{and} \quad 1-p^* = 0.4749$$

Usually we need the values of u and d so that we can fill in the future underlying asset prices in the tree. Since we are given the complete tree here, we don't actually need u and d. So, alternatively, we can calculate p^* directly from the entries in the tree, ie $p^* = \frac{50e^{0.01} - 40}{60-40}$, which gives the same answer. This is just the $p^* = \frac{e^{(r-\delta)h} - d}{u-d}$ formula, but with each term multiplied by 50.

Similarly, for the 60-70-50 branch, we have:

$$p^* = \frac{60e^{0.01} - 50}{70-50} = 0.5302 \quad \text{and} \quad 1-p^* = 0.4698$$

We don't need to find the risk-neutral probabilities for the 40-50-30 branch since the payoffs there are both zero.

We can now calculate the option prices in the usual way, as discounted risk-neutral expectations. The option price at the 60 node is:

$$e^{-0.01}\left[0.5302 \times 10 + 0.4698 \times 0\right] = 5.25$$

The initial option price is:

$$e^{-0.01}\left[0.5251 \times 5.25 + 0.4749 \times 0\right] = 2.73$$