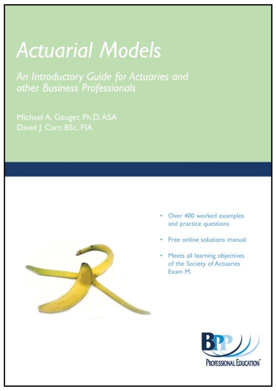

Actuarial models

By Michael A Gauger

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Solutions to practice questions – Chapter 3

Solution 3.1

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0,1,2 \\ \ddot{a}_{\overline{3}|} & \text{for } K \geq 3 \end{cases}, \text{ where } K = K(30).$$

Solution 3.2

$$\begin{aligned} E[Y] &= \ddot{a}_{\overline{1}|}\, q_{30} + \ddot{a}_{\overline{2}|}\, {}_{1|}q_{30} + \ddot{a}_{\overline{3}|}\, {}_2p_{30} \\ &= 1 + vp_{30} + v^2\, {}_2p_{30} \end{aligned}$$

Solution 3.3

The current payment method is probably the simplest to use. From the given mortality rates, we have:

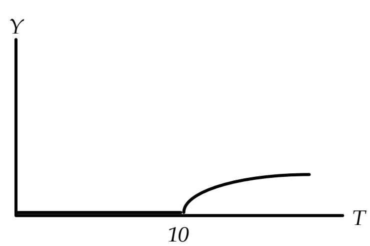
$$p_{30} = 1 - 0.010 = 0.99 \qquad {}_2p_{30} = (1 - q_{30})(1 - q_{31}) = 0.99 \times 0.985 = 0.97515$$

So we must have:

$$E[Y] = 1 + vp_{30} + v^2\, {}_2p_{30} = 1 + \frac{0.99}{1.05} + \frac{0.97515}{1.05^2} = 2.82735$$

Solution 3.4

$$Y = \begin{cases} 0 & T \leq 10 \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{10}|} & T > 10 \end{cases} \quad \text{where } T = T(55)$$



Solution 3.5

$$\begin{aligned}1000\,{}_{10|}\bar{a}_{55} &= 1000\left(\int_0^{10} 0\, f_{T(55)}(t)\,dt + \int_{10}^{\omega-55}\left(\bar{a}_{\overline{t}|} - \bar{a}_{\overline{10}|}\right) f_{T(55)}(t)\,dt\right)\\ &= 1000\int_{10}^{\omega-55} v^t\,{}_tp_{55}\,dt\end{aligned}$$

Solution 3.6

Since $\delta = 0.05$, we have $v^t = e^{-0.05t}$. With de Moivre's law, the future lifetime of (55) is uniformly distributed on the interval $(0, 35]$. The survival function is thus: ${}_tp_{55} = 1 - t/35$ for $0 < t \le 35$. Evaluating the current payment formula will require an integration by parts to evaluate $\int_{10}^{35} t\,e^{-0.05t}\,dt$. However, evaluating the aggregate payment formula will only require exponential integrals.

$$\begin{aligned}1000\,{}_{10|}\bar{a}_{55} &= 1000\int_{10}^{\omega-55}\left(\bar{a}_{\overline{t}|} - \bar{a}_{\overline{10}|}\right) f_{T(55)}(t)\,dt\\ &= 1000\int_{10}^{35}\left(\frac{1-e^{-0.05t}}{.05} - 7.86939\right)\frac{1}{35}\,dt\\ &= \frac{1000}{35}\left(20\int_{10}^{35} dt - 20\int_{10}^{35} e^{-0.05t}\,dt - 7.86939\int_{10}^{35} dt\right)\\ &= \frac{1000}{35}\left(20\times 25 + 20\left(\frac{e^{-0.05t}}{0.05}\Bigg|_{10}^{35}\right) - 7.86939\times 25\right)\\ &= 3{,}719\end{aligned}$$

Solution 3.7

The fund is $1.1\times 3{,}719 = 4{,}091$ to the nearest dollar. The life (55) will have to live at least 10 years for Y to exceed this number. So we have:

$$\begin{aligned}\Pr(Y \le F) &= 1 - \Pr(Y > 4{,}091) = 1 - \Pr\left(\bar{a}_{\overline{T}|} - \bar{a}_{\overline{10}|} > 4.091\right)\\ &= 1 - \Pr\left(\bar{a}_{\overline{T}|} > 11.960\right) = 1 - \Pr\left(\frac{1-e^{-0.05T}}{0.05} > 11.960\right)\\ &= 1 - \Pr\left(0.40199 > e^{-0.05T}\right) = 1 - \Pr(T > 18.227)\\ &= 1 - {}_{18.227}p_{55} = 1 - \left(1 - \frac{18.227}{35}\right) = 0.521\end{aligned}$$

Solution 3.8

$$\begin{aligned}\ddot{a}_x &= 1 + vp_x + v^2{}_2p_x + \cdots \\ &= 1 + e^{-\delta}e^{-\mu} + e^{-2\delta}\,e^{-2\mu} + \cdots \quad \text{(geometric series)} \\ &= \frac{1}{1-e^{-(\delta+\mu)}} = \frac{e^{\delta}}{e^{\delta}-e^{-\mu}} = \frac{1+i}{1+i-(1-q_x)} = \frac{1+i}{i+q_x}\end{aligned}$$

Solution 3.9

Y is a discrete random variable with only 3 possible values:

$$\begin{aligned}&Y = 1 \text{ if } K = 0 \text{ and } \Pr(K=0) = q_{30} = 0.01 \\ &Y = 1 + v = 1.95238 \text{ if } K = 1 \text{ and } \Pr(K=1) = {}_{1|}q_{30} = 0.99 \times 0.015 = 0.01485 \\ &Y = 1 + v + v^2 = 2.85941 \text{ if } K \ge 2 \text{ and } \Pr(K \ge 2) = 0.99 \times 0.985 = 0.97515\end{aligned}$$

We already have $E[Y] = 2.82735$. The second moment is:

$$\begin{aligned}E\left[Y^2\right] &= 1^2 \times 0.01 + (1.95238)^2 \times 0.01485 + (2.8591)^2 \times 0.97515 \\ &= 8.03965\end{aligned}$$

So the variance is $8.03965 - 2.82735^2 = 0.04576$.

Solution 3.10

There is a standard formula for this variance:

$$\text{var}(Y) = 1{,}000^2 \times \frac{{}^2\bar{A}_{55} - \left(\bar{A}_{55}\right)^2}{\delta^2}$$

There is also a shortcut calculation of insurance moments for de Moivre's law (see Section 2.6 of Chapter 2)":

$$\begin{aligned}\bar{A}_x &= \frac{\bar{a}_{\overline{\omega-x}|}}{\omega - x} \quad \Rightarrow \\ \bar{A}_{55} &= \frac{\bar{a}_{\overline{35}|}}{35} = \frac{1-e^{-35\times0.05}}{35\times0.05} = 0.47213 \\ {}^2\bar{A}_{55} &= \frac{{}^2\bar{a}_{\overline{35}|}}{35} = \frac{1-e^{-35\times0.10}}{35\times0.10} = 0.27709\end{aligned}$$

Plug these numbers into the variance formula above and the resulting variance is 21,672,202.

Solution 3.11

The aggregate present value is $S = Y_1 + Y_2 + \cdots + Y_{100}$ where the Y_i are independent. In Solution 3.3 we saw that $E[Y] = 2.82735$. In Solution 3.9 we calculated $\text{var}(Y) = 0.04576$. For the distribution of *S*, we have:

$$E[S] = 100\,E[Y] = 282.735 \qquad \text{var}(S) = 100\,\text{var}(Y) = 4.576$$

The fund *F* is $1.1 \times 282.735 = 311.009$. Since this number is approximately 13 standard deviations above the mean, it is virtually certain that the fund is sufficient. The probability of insufficiency is virtually 0.

Solution 3.12

$$100\ddot{a}^{(2)}_{30:\overline{3}|} = 50 + 50v^{0.5}\,{}_{0.5}p_{30} + 50\,v\,p_{30} + 50\,v^{1.5}\,{}_{1.5}p_{30} + 50v^2\,{}_2p_{30} + 50\,v^{2.5}\,{}_{2.5}p_{30}$$

To evaluate this formula, use $v = 1/1.05$, and the UDD rule ${}_tp_x = 1 - t\,q_x$ when x is a whole age and t is a fractional part of a year:

$$100\ddot{a}^{(2)}_{30:\overline{3}|} = 50 + \frac{50 \times (1 - 0.5 \times 0.01)}{1.05^{0.5}} + \frac{50 \times 0.99}{1.05} + \frac{50 \times 0.99 \times (1 - 0.5 \times 0.015)}{1.05^{1.5}}$$

$$+ \frac{50 \times 0.99 \times 0.985}{1.05^2} + \frac{50 \times 0.99 \times 0.985 \times (1 - 0.5 \times 0.020)}{1.05^{2.5}} = 278.31$$

Solution 3.13

Use the insurance shortcut for de Moivre's law:

$$A_{60} = \frac{a_{\overline{30}|}}{30} = \frac{1 - 1.06^{-30}}{30 \times 0.06} = 0.45883 \qquad A_{75} = \frac{a_{\overline{15}|}}{15} = \frac{1 - 1.06^{-15}}{15 \times 0.06} = 0.64748$$

Using the annuity-insurance relation, the value of the whole life annuity due is:

$$1{,}000\ddot{a}_{60} = 1{,}000 \times \frac{1 - A_{60}}{d} = 9{,}560.71$$

The value of the certain and life annuity due is:

$$APV = 1{,}000(\ddot{a}_{\overline{15}|} + {}_{15|}\ddot{a}_{60}) = 1{,}000\left(\ddot{a}_{\overline{15}|} + v^{15}\,{}_{15}p_{60}\,\ddot{a}_{75}\right)$$

$$= 1{,}000\left(\frac{1 - 1.06^{-15}}{0.06/1.06} + 1.06^{-15} \times \frac{15}{30} \times \frac{1 - 0.64748}{0.06/1.06}\right) = 11{,}594.30$$

This is approximately a 21.3% increase in the APV.

Solution 3.14

Note first that we have:

$$d^{(2)} = 2\left(1 - v^{0.5}\right) = 2\left(1 - 1.06^{-0.5}\right) = 0.05743$$

Now let's do the APV calculation:

$$\begin{aligned} APV &= 1,000\left(\ddot{a}^{(2)}_{\overline{10|}} + {}_{10|}\ddot{a}^{(2)}_{40}\right) = 1,000\left(\frac{1 - v^{10}}{d^{(2)}} + v^{10}\,{}_{10}p_{40}\,\ddot{a}^{(2)}_{50}\right) \\ &= 1,000\left(\frac{1 - 1.06^{-10}}{0.05743} + 0.53667 \times \left(\alpha(2)\ddot{a}_{50} - \beta(2)\right)\right) \\ &= 1,000\left(7.68968 + 0.53667 \times \left(1.00021 \times 13.2668 - 0.25739\right)\right) = 14,673 \end{aligned}$$

Solution 3.15

$K(62)$	Probability	Y
0	0.02	50
1	0.04 – 0.02 = 0.02	$50 + 75v = 121.42857$
≥ 2	0.96	$50 + 75v + 100v^2 = 212.13152$

Solution 3.16

From the table we have:

$$\begin{aligned} E[Y] &= 50 \times 0.02 + 121.42857 \times 0.02 + 212.13152 \times 0.96 = 207.07483 \\ E\left[Y^2\right] &= 50^2 \times 0.02 + 121.42857^2 \times 0.02 + 212.13152^2 \times 0.96 = 43,544.69 \\ \text{var}(Y) &= 664.70 \end{aligned}$$

Solution 3.17

With de Moivre's law we have $q_x = 1/(\omega - x) = 1/(100 - x)$. The backward recursion formula is:

$$A_x = vq_x + v\,p_x\,A_{x+1} = \frac{q_x + p_x\,A_{x+1}}{1.05}$$

The starting point is $A_{100} = 1$. The recursion formula leads to:

$$A_{99} = \frac{\frac{1}{1} + 0 \times A_{100}}{1.05} = 0.95238 \qquad A_{98} = \frac{\frac{1}{2} + \frac{1}{2} \times 0.95238}{1.05} = 0.92971$$

$$A_{97} = \frac{\frac{1}{3} + \frac{2}{3} \times 0.92971}{1.05} = 0.90775 \qquad A_{96} = \frac{\frac{1}{4} + \frac{3}{4} \times 0.90775}{1.05} = 0.88649$$

Solution 3.18

You should first notice that p_{81} is also 0.90.

$$e_x = p_x + p_x\,e_{x+1} \;\Rightarrow\; e_{x+1} = \frac{e_x - p_x}{p_x}$$

$$e_{81} = \frac{8.5 - 0.9}{0.9} = 8.444 \qquad e_{82} = \frac{8.444 - 0.9}{0.9} = 8.383$$

Solution 3.19

$$F = 1.2 \int_0^5 \underbrace{100(1 - 0.2w)}_{\text{amount}} \times \underbrace{e^{-0.05w}}_{\text{discount}} \times \underbrace{0.10e^{-0.10w}\,dw}_{\text{probability } f(w)\,dw}$$

$$= 12\left(\int_0^5 e^{-0.15w}\,dw - 0.2\int_0^5 we^{-0.15w}\,dw\right) = 12\left(\left(-\frac{e^{-0.15w}}{0.15}\bigg|_0^5 - 0.2\left(-\frac{e^{-0.05w}(0.15w+1)}{0.15^2}\bigg|_0^5\right)\right)\right) =$$

$$= 12(3.51756 - 1.54096) = 23.71910$$

Solution 3.20

(i) $1{,}000\, A_{45:\overline{5}|}^{\;\;1} = 1{,}000\, v^5\, {}_5p_{45} = \dfrac{1{,}000\, l_{50}}{1.06^5\, l_{45}} = 729.88$

(ii) $\ddot{a}_{45:\overline{5}|} = \ddot{a}_{45} - v^5\, {}_5p_{45}\, \ddot{a}_{50} = 14.1121 - \left(\dfrac{729.88}{1{,}000}\right) 13.2668 = 4.42896$

(iii) $\bar{a}_{40:\overline{10}|} = \dfrac{1 - \bar{A}_{40:\overline{10}|}}{\delta} = \dfrac{1 - \left(\dfrac{i}{\delta} A^1_{40:\overline{10}|} + A_{40:\overline{10}|}^{\;\;\;1}\right)}{\ln(1.06)}$

$$= \frac{1 - \left(\dfrac{0.06}{\ln(1.06)}\left(0.16132 - 0.53667 \times 0.24905\right) + 0.53667\right)}{\ln(1.06)}$$

$$= 7.46274$$