

Actuarial models

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Solutions to practice questions – Chapter 3

Solution 3.1

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{for } K = 0, 1, 2 \\ \ddot{a}_{\overline{3}|} & \text{for } K \geq 3 \end{cases}, \text{ where } K = K(30).$$

Solution 3.2

$$\begin{aligned} E[Y] &= \ddot{a}_{\overline{1}|} q_{30} + \ddot{a}_{\overline{2}|} {}_1|q_{30} + \ddot{a}_{\overline{3}|} {}_2p_{30} \\ &= 1 + vp_{30} + v^2 {}_2p_{30} \end{aligned}$$

Solution 3.3

The current payment method is probably the simplest to use. From the given mortality rates, we have:

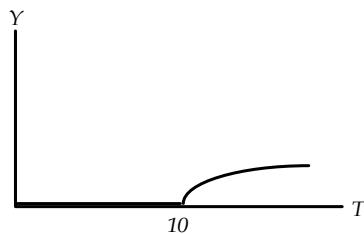
$$p_{30} = 1 - 0.010 = 0.99 \quad {}_2p_{30} = (1 - q_{30})(1 - q_{31}) = 0.99 \times 0.985 = 0.97515$$

So we must have:

$$E[Y] = 1 + vp_{30} + v^2 {}_2p_{30} = 1 + \frac{0.99}{1.05} + \frac{0.97515}{1.05^2} = 2.82735$$

Solution 3.4

$$Y = \begin{cases} 0 & T \leq 10 \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{10}|} & T > 10 \end{cases} \text{ where } T = T(55)$$



Solution 3.5

$$\begin{aligned} 1000 {}_{10|}\bar{a}_{55} &= 1000 \left(\int_0^{10} {}_0f_{T(55)}(t) dt + \int_{10}^{\omega-55} (\bar{a}_{\overline{t}|} - \bar{a}_{\overline{10}|}) f_{T(55)}(t) dt \right) \\ &= 1000 \int_{10}^{\omega-55} v^t {}_t p_{55} dt \end{aligned}$$

Solution 3.6

Since $\delta = 0.05$, we have $v^t = e^{-0.05t}$. With de Moivre's law, the future lifetime of (55) is uniformly distributed on the interval $(0, 35]$. The survival function is thus: ${}_t p_{55} = 1 - t/35$ for $0 < t \leq 35$. Evaluating the current payment formula will require an integration by parts to evaluate $\int_{10}^{35} t e^{-0.05t} dt$. However, evaluating the aggregate payment formula will only require exponential integrals.

$$\begin{aligned} 1000 {}_{10|}\bar{a}_{55} &= 1000 \int_{10}^{\omega-55} (\bar{a}_{\overline{t}|} - \bar{a}_{\overline{10}|}) f_{T(55)}(t) dt \\ &= 1000 \int_{10}^{35} \left(\frac{1 - e^{-0.05t}}{0.05} - 7.86939 \right) \frac{1}{35} dt \\ &= \frac{1000}{35} \left(20 \int_{10}^{35} dt - 20 \int_{10}^{35} e^{-0.05t} dt - 7.86939 \int_{10}^{35} dt \right) \\ &= \frac{1000}{35} \left(20 \times 25 + 20 \left(\frac{e^{-0.05t}}{0.05} \right) \Big|_{10}^{35} - 7.86939 \times 25 \right) \\ &= 3,719 \end{aligned}$$

Solution 3.7

The fund is $1.1 \times 3,719 = 4,091$ to the nearest dollar. The life (55) will have to live at least 10 years for Y to exceed this number. So we have:

$$\begin{aligned} \Pr(Y \leq F) &= 1 - \Pr(Y > 4,091) = 1 - \Pr(\bar{a}_{\overline{T}|} - \bar{a}_{\overline{10}|} > 4.091) \\ &= 1 - \Pr(\bar{a}_{\overline{T}|} > 11.960) = 1 - \Pr\left(\frac{1 - e^{-0.05T}}{0.05} > 11.960\right) \\ &= 1 - \Pr(0.40199 > e^{-0.05T}) = 1 - \Pr(T > 18.227) \\ &= 1 - {}_{18.227}p_{55} = 1 - \left(1 - \frac{18.227}{35}\right) = 0.521 \end{aligned}$$

Solution 3.8

$$\begin{aligned}\ddot{a}_x &= 1 + vp_x + v^2 {}_2p_x + \dots \\ &= 1 + e^{-\delta} e^{-\mu} + e^{-2\delta} e^{-2\mu} + \dots \quad (\text{geometric series}) \\ &= \frac{1}{1 - e^{-(\delta+\mu)}} = \frac{e^\delta}{e^\delta - e^{-\mu}} = \frac{1+i}{1+i-(1-q_x)} = \frac{1+i}{i+q_x}\end{aligned}$$

Solution 3.9

Y is a discrete random variable with only 3 possible values:

$$Y=1 \text{ if } K=0 \text{ and } \Pr(K=0)=q_{30}=0.01$$

$$Y=1+v=1.95238 \text{ if } K=1 \text{ and } \Pr(K=1)={}_1|q_{30}=0.99 \times 0.015 = 0.01485$$

$$Y=1+v+v^2=2.85941 \text{ if } K \geq 2 \text{ and } \Pr(K \geq 2)=0.99 \times 0.985 = 0.97515$$

We already have $E[Y]=2.82735$. The second moment is:

$$\begin{aligned}E[Y^2] &= 1^2 \times 0.01 + (1.95238)^2 \times 0.01485 + (2.85941)^2 \times 0.97515 \\ &= 8.03965\end{aligned}$$

So the variance is $8.03965 - 2.82735^2 = 0.04576$.

Solution 3.10

There is a standard formula for this variance:

$$\text{var}(Y) = 1,000^2 \times \frac{{}^2\bar{A}_{55} - (\bar{A}_{55})^2}{\delta^2}$$

There is also a shortcut calculation of insurance moments for de Moivre's law (see Section 2.6 of Chapter 2)':

$$\begin{aligned}\bar{A}_x &= \frac{\bar{a}_{\omega-x}}{\omega-x} \Rightarrow \\ \bar{A}_{55} &= \frac{\bar{a}_{35}}{35} = \frac{1 - e^{-35 \times 0.05}}{35 \times 0.05} = 0.47213 \\ {}^2\bar{A}_{55} &= \frac{{}^2\bar{a}_{35}}{35} = \frac{1 - e^{-35 \times 0.10}}{35 \times 0.10} = 0.27709\end{aligned}$$

Plug these numbers into the variance formula above and the resulting variance is 21,672,202.

Solution 3.11

The aggregate present value is $S=Y_1+Y_2+\dots+Y_{100}$ where the Y_i are independent. In Solution 3.3 we saw that $E[Y]=2.82735$. In Solution 3.9 we calculated $\text{var}(Y)=0.04576$. For the distribution of S , we have:

$$E[S]=100E[Y]=282.735 \quad \text{var}(S)=100\text{var}(Y)=4.576$$

The fund F is $1.1 \times 282.735 = 311.009$. Since this number is approximately 13 standard deviations above the mean, it is virtually certain that the fund is sufficient.

Solution 3.12

$$100\ddot{a}_{30:\overline{3}|}^{(2)} = 50 + 50v^{0.5} {}_{0.5}p_{30} + 50vp_{30} + 50v^{1.5} {}_{1.5}p_{30} + 50v^2 {}_2p_{30} + 50v^{2.5} {}_{2.5}p_{30}$$

To evaluate this formula, use $v=1/1.05$, and the UDD rule ${}_t p_x = 1 - tq_x$ when x is a whole age and t is a fractional part of a year:

$$\begin{aligned} 100\ddot{a}_{30:\overline{3}|}^{(2)} &= 50 + \frac{50 \times (1 - 0.5 \times 0.01)}{1.05^{0.5}} + \frac{50 \times 0.99}{1.05} + \frac{50 \times 0.99 \times (1 - 0.5 \times 0.015)}{1.05^{1.5}} \\ &\quad + \frac{50 \times 0.99 \times 0.985}{1.05^2} + \frac{50 \times 0.99 \times 0.985 \times (1 - 0.5 \times 0.020)}{1.05^{2.5}} = 278.31 \end{aligned}$$

Solution 3.13

Use the insurance shortcut for de Moivre's law:

$$A_{60} = \frac{a_{\overline{30}|}}{30} = \frac{1 - 1.06^{-30}}{30 \times 0.06} = 0.45883 \quad A_{75} = \frac{a_{\overline{15}|}}{15} = \frac{1 - 1.06^{-15}}{15 \times 0.06} = 0.64748$$

Using the annuity-insurance relation, the value of the whole life annuity due is:

$$1,000\ddot{a}_{60} = 1,000 \times \frac{1 - A_{60}}{d} = 9,560.71$$

The value of the certain and life annuity due is:

$$\begin{aligned} APV &= 1,000(\ddot{a}_{\overline{15}|} + {}_{15|}\ddot{a}_{60}) = 1,000\left(\ddot{a}_{\overline{15}|} + v^{15} {}_{15}p_{60} \ddot{a}_{75}\right) \\ &= 1,000 \left(\frac{1 - 1.06^{-15}}{0.06/1.06} + 1.06^{-15} \times \frac{15}{30} \times \frac{1 - 0.64748}{0.06/1.06} \right) = 11,594.30 \end{aligned}$$

This is approximately a 21.3% increase in the APV.

Solution 3.14

Note first that we have:

$$d^{(2)} = 2(1 - v^{0.5}) = 2(1 - 1.06^{-0.5}) = 0.05743$$

Now let's do the APV calculation:

$$\begin{aligned} APV &= 1,000 \left(\ddot{a}_{\overline{10}|}^{(2)} + {}_{10|}\ddot{a}_{40}^{(2)} \right) = 1,000 \left(\frac{1 - v^{10}}{d^{(2)}} + v^{10} {}_{10}p_{40} \ddot{a}_{50}^{(2)} \right) \\ &= 1,000 \left(\frac{1 - 1.06^{-10}}{0.05743} + 0.53667 \times (\alpha(2)\ddot{a}_{50} - \beta(2)) \right) \\ &= 1,000 (7.68968 + 0.53667 \times (1.00021 \times 13.2668 - 0.25739)) = 14,673 \end{aligned}$$

Solution 3.15

K(62)	Probability	Y
0	0.02	50
1	0.04 - 0.02 = 0.02	50 + 75v = 121.42857
≥2	0.96	50 + 75v + 100v ² = 212.13152

Solution 3.16

From the table we have:

$$E[Y] = 50 \times 0.02 + 121.42857 \times 0.02 + 212.13152 \times 0.96 = 207.07483$$

$$E[Y^2] = 50^2 \times 0.02 + 121.42857^2 \times 0.02 + 212.13152^2 \times 0.96 = 43,544.69$$

$$\text{var}(Y) = 664.70$$

Solution 3.17

With de Moivre's law we have $q_x = 1/(\omega - x) = 1/(100 - x)$. The backward recursion formula is:

$$A_x = vq_x + v p_x A_{x+1} = \frac{q_x + p_x A_{x+1}}{1.05}$$

The starting point is $A_{100} = 1$. The recursion formula leads to:

$$\begin{aligned} A_{99} &= \frac{\frac{1}{1} + 0 \times A_{100}}{1.05} = 0.95238 & A_{98} &= \frac{\frac{1}{2} + \frac{1}{2} \times 0.95238}{1.05} = 0.92971 \\ A_{97} &= \frac{\frac{1}{3} + \frac{2}{3} \times 0.92971}{1.05} = 0.90775 & A_{96} &= \frac{\frac{1}{4} + \frac{3}{4} \times 0.90775}{1.05} = 0.88649 \end{aligned}$$

Solution 3.18

You should first notice that p_{81} is also 0.90.

$$\begin{aligned} e_x &= p_x + p_x e_{x+1} \Rightarrow e_{x+1} = \frac{e_x - p_x}{p_x} \\ e_{81} &= \frac{8.5 - 0.9}{0.9} = 8.444 & e_{82} &= \frac{8.444 - 0.9}{0.9} = 8.383 \end{aligned}$$

Solution 3.19

$$\begin{aligned} F &= 1.2 \int_0^5 \underbrace{100(1-0.2w)}_{\text{amount}} \times \underbrace{e^{-0.05w}}_{\text{discount}} \times \underbrace{0.10e^{-0.10w} dw}_{\text{probability } f(w)dw} \\ &= 12 \left(\int_0^5 e^{-0.15w} dw - 0.2 \int_0^5 w e^{-0.15w} dw \right) = 12 \left(\left(-\frac{e^{-0.15w}}{0.15} \right) \Big|_0^5 - 0.2 \left(-\frac{e^{-0.05w} (0.15w + 1)}{0.15^2} \right) \Big|_0^5 \right) \\ &= 12(3.51756 - 1.54096) = 23.71910 \end{aligned}$$

Solution 3.20

$$(i) \quad 1,000 A_{45:\overline{5}|}^1 = 1,000 v^5 {}_5p_{45} = \frac{1,000 l_{50}}{1.06^5 l_{45}} = 729.88$$

$$(ii) \quad \ddot{a}_{45:\overline{5}|} = \ddot{a}_{45} - v^5 {}_5p_{45} \ddot{a}_{50} = 14.1121 - \left(\frac{729.88}{1,000} \right) 13.2668 = 4.42896$$

$$(iii) \quad \bar{a}_{40:\overline{10}|} = \frac{1 - \bar{A}_{40:\overline{10}|}}{\delta} = \frac{1 - \left(\frac{i}{\delta} A_{40:\overline{10}|}^1 + A_{40:\overline{10}|}^1 \right)}{\ln(1.06)}$$

$$= \frac{1 - \left(\frac{0.06}{\ln(1.06)} (0.16132 - 0.53667 \times 0.24905) + 0.53667 \right)}{\ln(1.06)}$$

$$= 7.46274$$