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1

Interest rates

Overview

In addition to being a medium of exchange, money can also be thought of as a cash amount that is invested to earn even more cash as **interest**. The word interest can be translated from the Greek language as the birth of money from money.

Interest is defined as the payment by one party (the borrower) for the use of an asset that belongs to another party (the lender) over a period of time. This asset is also known as **capital**. In this sense, capital is just money that earns interest. Other forms of productive capital exist, but with interest theory, we are primarily interested in monetary capital, which we also refer to as the **principal** amount.

When interest is expressed as a percent of the capital amount, it is referred to as an **interest rate**. Interest rates are most often computed on an annual basis, but they can be determined for non-annual time periods as well.

We have two perspectives to consider. The first is the perspective of the owner of the capital who would like to be compensated for lending it. The second is the perspective of the borrower of the capital who is willing to pay the lender for the right to borrow it.

The borrower pays interest to the lender for the use of the money. The lender may require additional compensation for the risk of **default**, which is the risk that the borrower will not be able to repay the loan principal. If some risk of default exits, the lender will typically demand a higher interest rate to compensate for assuming this risk of default.

Lenders typically require **collateral**, *ie* something of value pledged as security against the risk of default, to help guarantee the loan will be repaid. The comedian Bob Hope once observed that "A bank is a place that will lend you money if you can prove you don't need it."

In this first chapter, we consider interest earned over annual time periods, but in later chapters we consider non-annual time periods as well. We look at five ways to express interest: **simple interest**, **compound interest**, **simple discount**, **compound discount**, and **continuous interest**. We can break these methods down into two classes: simplistic and realistic.

The simple interest and simple discount methods, as their names imply, belong to the simplistic class. These methods are impractical for large, complicated financial transactions. They may be used, however, for small, simple transactions.

The compound interest, compound discount, and continuous interest methods belong to the realistic class. Compound interest is generally used for financial transactions. Compound discount is less common, but it isn't difficult to grasp once we become comfortable with compound interest. Continuous interest appears frequently in academic and theoretical discussions of valuation.

All three of the realistic methods are consistent with one another. Just as distance can be measured in miles or kilometers, interest can be measured using compound interest, compound discount, or continuous interest rates. And just as a European visiting America could insist on converting all distance measurements into metric, a finance professor working at a bank could insist on converting compound interest rates into continuous interest rates. The professor's calculations would produce the correct values regardless of which type of rate was used.

Compound interest, which allows interest to grow on previously earned interest, has amazing accumulation powers when compared to simple interest, which does not allow interest to grow on previously earned interest. Albert Einstein is said to have noted that the most powerful force in the universe is compound interest.

In this chapter, we also develop methods for **accumulating** payments to a future point in time and **discounting** payments to a previous point in time.

When a payment is accumulated to a future point in time, its **future value** is calculated, taking into account any interest that will be earned during the investment period. When a payment is discounted to a previous point in time, its **present value** is calculated, taking into account any interest that will be earned during the investment period. In order to value a cash flow, we need to know the exact amount of the cash flow and the exact timing of the cash flow. This chapter illustrates the **time value of money**, *ie* a $1 payment now is worth more than $1 payable in one year's time.

1.1 Interest

Interest on savings accounts

When money is deposited into a bank account, it typically earns interest. The depositor can be thought of as the **lender** and the bank can be thought of as the **borrower**. The borrower pays interest to the lender to compensate for the use of the money and any risk of default.

Let's say a person deposits $1,000 into a bank account. One year later, the account has accumulated to $1,050. This amount consists of $1,000, representing the initial deposit or **capital**, and $50, representing the **interest** earned on the deposit over the year. The amount of interest earned over a period of time is simply the difference between the accumulated account value at the end of the period and the accumulated account value at the beginning of the period.

For any amount of interest earned over a given period, we can also calculate the associated **interest rate**. The interest rate in effect for a one-year period is the amount of interest earned over the year divided by the initial accumulated value.

In this example, the interest rate for the year is:

$$\frac{1{,}050-1{,}000}{1{,}000}=\frac{50}{1{,}000}=0.05 \quad \text{or} \quad 5\%$$

At this point, let's keep things simple by considering annual time periods only, but in later chapters we will consider non-annual time periods.

Interest

The amount of interest earned from time t to time $t+s$ is:

$$AV_{t+s}-AV_t$$

where AV_t is the accumulated value at time t.

Interest rate

The annual interest rate i in effect from time t to time $t+1$ is:

$$i=\frac{AV_{t+1}-AV_t}{AV_t}$$

where t is measured in years.

For example, Bob invests $3,200 in a savings account on January 1, 2004. On December 31, 2004, the account balance has grown to $3,294.08.

The total interest earned during 2004 is:

$$3{,}294.08-3{,}200.00=\$94.08$$

The annual interest rate during 2004 is:

$$\frac{3{,}294.08-3{,}200.00}{3{,}200.00}=\frac{94.08}{3{,}200.00}=0.0294 \quad \text{or} \quad 2.94\%$$

Interest on loans

The same theory and definitions can be applied to a loan.

A person may borrow money from a bank, *eg* to buy a car. In this case, the bank is the lender, and the individual is the borrower. As before, the borrower pays interest to the lender to compensate for the use of the money and any risk of default.

Let's say a person borrows $12,000 from a bank. The loan is to be repaid in full in one year's time with a payment to the bank of $12,780. This amount consists of $12,000, representing the initial amount of the loan, and $780, representing the interest paid on the loan over the year.

The annual interest rate in this example is:

$$\frac{12{,}780-12{,}000}{12{,}000}=\frac{780}{12{,}000}=0.065 \quad \text{or} \quad 6.5\%$$

When we use interest rates in the context of a loan, two important points should be kept in mind:

1. The **principal amount** of the loan is the amount provided to the borrower when the loan is originated. (In the example above, the principal amount is $12,000.)

2. Interest begins to accrue when the loan is originated and continues to accrue until the loan is repaid in full.

The second point is analogous to a savings account: interest begins to accrue when the money is deposited and continues to accrue until the deposit is repaid (withdrawn) in full.

1.2 Simple interest

When money is invested in an account paying **simple interest**, interest is only earned on the initial deposit. Interest is not earned on the interest that has previously accrued.

Simple interest

If $\$X$ is invested in an account that pays simple interest at a rate of i per year, then the accumulated value of the investment after t years is:

$$AV_t = X(1+ti)$$

The accumulated value is composed of the initial investment of $\$X$ and t years of simple interest at the rate of i per year, $\$Xti$. The interest earned each year is $\$Xi$.

For example, let's assume a bank account is opened with a deposit of $100. The account pays simple interest of 8% per year. We have $i = 0.08$ and $X = 100$.

The accumulated value of the account at the end of one year is:

$$AV_1 = 100(1+1\times 0.08) = \$108.00$$

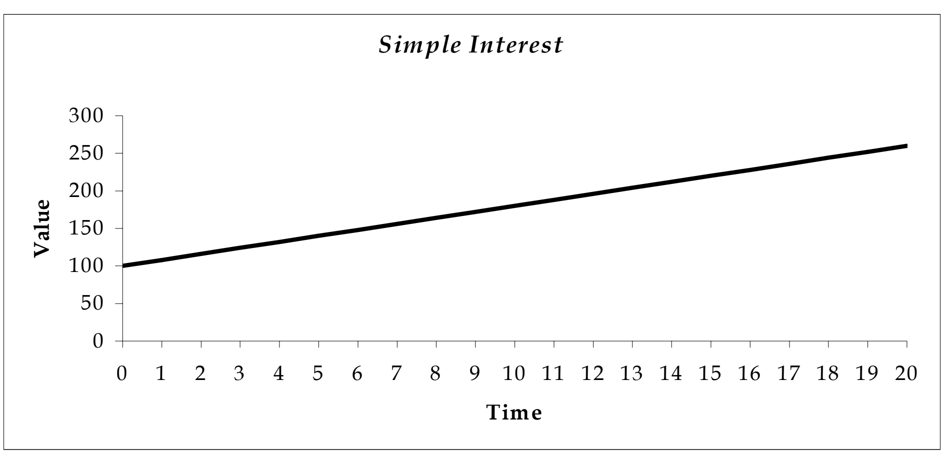
The accumulated value of the account at the end of ten years is:

$$AV_{10} = 100(1+10\times 0.08) = \$180.00$$

The accumulated value of the account at the end of twenty years is:

$$AV_{20} = 100(1+20\times 0.08) = \$260.00$$

With simple interest, the account value increases linearly over time. This is shown in the following graph.



As the continuous nature of the graph implies, the formula for the accumulated value of a deposit under simple interest still applies if t is not an integer. When t is not an integer, interest is paid on a pro-rata (proportional) basis.

For example, a bank account that pays simple interest of 6% per year is opened with a deposit of \$100. What is the accumulated value of the account at the end of 9 months?

Since we're working with an annual simple interest rate, t should be expressed in terms of the number of years on deposit. So we have $X = 100$, $t = 9/12 = 0.75$ years, and $i = 0.06$.

Hence, the accumulated value after 9 months is:

$$AV_{0.75} = 100(1 + 0.75 \times 0.06) = \$104.50$$

Example 1.1

A bank accounts pays 3.6% simple interest. Anna deposits \$10,000 on January 1, 2004 and leaves her funds to earn interest.

Calculate the accumulated value of Anna's account on April 1, 2006 and January 1, 2007.

Solution

On April 1, 2006, we have $t = 2 \text{ years } 3 \text{ months} = 2.25 \text{ years}$, so the accumulated value is:

$$AV_{2.25} = 10,000(1 + 2.25 \times 0.036) = \$10,810.00$$

On January 1, 2007, we have $t = 3$ years, so the accumulated value is:

$$AV_3 = 10,000(1 + 3 \times 0.036) = \$11,080.00$$

♦♦

Banks don't typically use simple interest because it is just too, well, simplistic. If banks actually paid simple interest, then depositors could earn more interest by pursuing the following simple strategy: as soon as interest is credited to the account, withdraw the total account value and immediately re-deposit it, using the interest paid-to-date to increase the size of the deposit. This is illustrated in the following example.

Example 1.2

A bank accounts pays 6% simple interest. Randy deposits \$100 and leaves his funds to earn interest for 2 years. Leonard also deposits \$100, but Leonard withdraws his accumulated value at the end of 1 year, and he then immediately returns the money to the bank, depositing it in a new account.

Who has the greater accumulated value at the end of 2 years: Randy or Leonard?

Solution

At the end of two years, Randy has \$112:

$$AV_2 = 100(1 + 2 \times 0.06) = \$112.00$$

At the end of 1 year, Leonard has \$106:

$$AV_1 = 100(1 + 1 \times 0.06) = \$106.00$$

But Leonard then withdraws the \$106 at the end of 1 year and deposits the \$106 in a new account, which also earns a simple interest rate of 6% per year. At the end of the second year, Leonard's accumulated value is \$112.36:

$$AV_2 = 106(1 + 1 \times 0.06) = \$112.36$$

Leonard has the greater accumulated value at the end of two years. ♦♦

In the example above, Leonard found a way to earn interest on the interest he earned in the first year. As we'll see in the next section, when interest is earned on interest, the interest is **compounding**.

If banks used simple interest, then depositors who withdrew and re-deposited their funds would have higher account values than the depositors who simply left their funds in their accounts. Since it doesn't make sense to reward depositors for withdrawing and re-depositing their funds, banks and other financial institutions don't actually use simple interest when calculating accumulated values.

However, simple interest is sometimes used in circumstances where accuracy is not very important. For example, if the time period is short or if the amount of money involved is small, then simple interest might be considered sufficiently accurate.

1.3 Compound interest

When money is deposited into an account paying **compound interest**, interest is earned on the initial deposit *and* the interest that has previously accrued. This is analogous to using simple interest, but periodically the interest is credited to the account and the interest rate then applies to the new, larger balance.

We examine annual compounding in this chapter, which means that interest is credited to the account annually. It is also possible to compound more or less frequently than annually, and we will learn more about non-annual compounding in Chapter 4.

Compound interest

If $\$X$ is invested in an account that pays compound interest at a rate of i per year, then the accumulated value of the investment after t years is:

$$AV_t = X(1+i)^t$$

Consider another bank account that is opened with a deposit of \$100. This account pays compound interest of 8% per year. Let's compare this compound interest account to the simple interest account that we examined earlier that paid simple interest of 8% per year.

Under both compound and simple interest, the accumulated value at the end of one year is \$108:

Compound interest account: $AV_1 = 100(1+0.08)^1 = \$108.00$

Simple interest account: $AV_1 = 100(1+1\times 0.08) = \$108.00$

With compound interest, the entire accumulated value, not just the original principal, earns interest. This causes the accumulated value in the second year to grow more quickly under compound interest than under simple interest. The accumulated value of the account under compound interest is \$0.64 greater at the end of two years than the accumulated value under simple interest:

Compound interest account: $AV_2 = 100(1+0.08)(1+0.08) = \$116.64$

Simple interest account: $AV_2 = 100(1+2\times 0.08) = \$116.00$

The \$0.64 difference is due to the fact that under compound interest, the \$8 of interest that was earned in the first year earns \$0.64 in interest over the second year:

$$8(0.08) = \$0.64$$

Instead of calculating the accumulated value year by year, we can use the formula from the box above to obtain the same accumulated value at the end of two years under compound interest:

$$AV_2 = 100(1+0.08)^2 = \$116.64$$

The accumulated value of the compound interest account at the end of ten years is:

$$AV_{10} = 100(1+0.08)^{10} = \$215.89$$

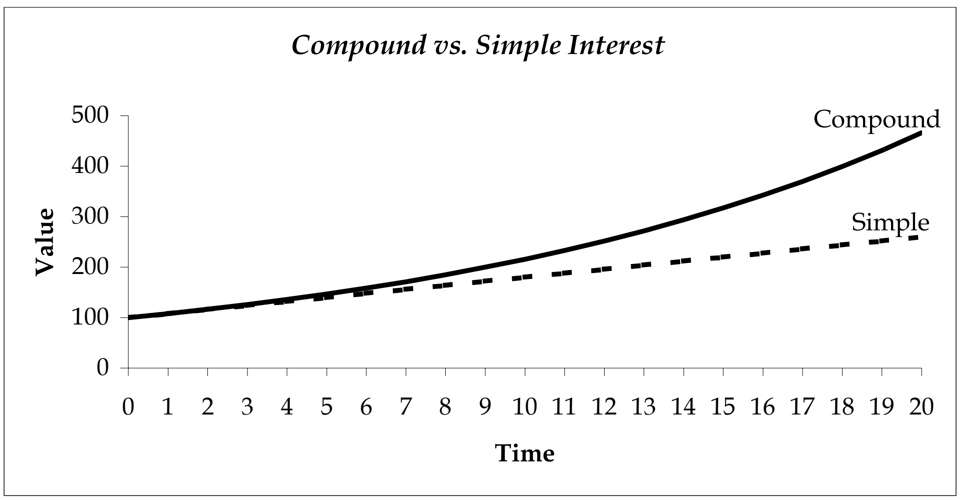
Notice that at the end of the tenth year, the accumulated value under compound interest exceeds the $180 accumulated value that we calculated under simple interest.

The accumulated value of the compound interest account at the end of the twentieth year is:

$$AV_{20} = 100(1+0.08)^{20} = \$466.10$$

At the end of the twentieth year, the accumulated value under compound interest is much greater than the $260 accumulated value under simple interest. This illustrates the power of compound interest. As time passes, an account paying compound interest exhibits geometric (or exponential) growth, while an account paying simple interest exhibits linear growth.

The following graph shows how the value of the account using compound interest increases over time. The dotted line shows the linear progression of the deposit earning simple interest.



As the graph implies, the formula for the accumulated value of a deposit under compound interest still applies if t is not an integer, assuming that interest is paid on a pro-rata basis.

For example, after 2.5 years the accumulated value of the compound interest account in the current example is:

$$AV_{2.5} = 100(1+0.08)^{2.5} = \$121.22$$

Example 1.3

Drew invests $100 on January 1, 2004 in a bank account that pays compound interest of 5% per year. What is the accumulated value of the account on October 1, 2005?

Solution

We have $X = 100$, $i = 0.05$, and $t = 1 \text{ year } 9 \text{ months} = 1.75 \text{ years}$.

So, the accumulated value of the account on October 1, 2005 is:

$$AV_{1.75} = 100(1+0.05)^{1.75} = \$108.91$$ ♦♦

Example 1.4

Helen borrows \$1,000 for 3 years at a compound interest rate of 11.65%. What will the total payment be in 3 years to repay both the loan principal and interest due on the loan?

Solution

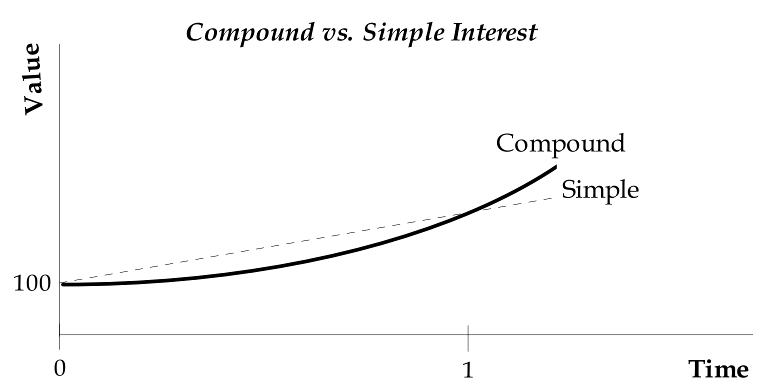
We have $X = 1{,}000$, $i = 0.1165$, and $t = 3$.

So, the total payment required in 3 years is:

$$AV_3 = 1{,}000(1.1165)^3 = \$1{,}391.80$$ ♦♦

Based on the discussion so far, one might think that the accumulated value of an account earning compound interest always exceeds the accumulated value of an account earning the same simple interest rate. But this is only true when the period is greater than one year. When the investment period is less than one year, the accumulated account value under simple interest actually exceeds the accumulated account value under the same compound interest rate.

Since the difference between the accumulated values is small when the elapsed time is less than one year, this effect is not easily observed in the previous chart. So, let's magnify the portion of the chart from time 0 to 1 year. The graph below illustrates that the accumulated value under compound interest is less than the accumulated value under simple interest during the first year.



Note: this graph is not drawn to scale.

For example, if the simple interest rate and the compound interest rate are each 5%, let's determine the accumulated amount of a \$100 deposit after 6 months.

Under simple interest and compound interest, the deposit accumulates to:

Compound interest account: $AV_{0.5} = 100(1+0.05)^{0.5} = \$102.47$

Simple interest account: $AV_{0.5} = 100(1+0.5\times 0.05) = \$102.50$

This may not seem like a big difference, but consider a large pension fund deposit of \$100,000,000. Instead of a \$0.03 difference between account values, we have a \$30,000 difference!

We can also state the relationship between simple and compound interest mathematically.

For $i > 0$, we have:

$1+ti > (1+i)^t$ for $0 < t < 1$ *ie* simple interest greater before one year

$1+ti = (1+i)^t$ for $t = 1$ *ie* simple and compound interest equal over one year

$1+ti < (1+i)^t$ for $t > 1$ *ie* compound interest greater after more than one year

Let's take another look at Example 1.2. This time, the bank credits interest to Leonard and Randy at a compound interest rate, and the resulting treatment is more equitable.

Example 1.5

A bank accounts pays 6% compound interest. Randy deposits \$100 and leaves his funds to earn interest for 2 years. Leonard also deposits \$100, but Leonard withdraws his accumulated value at the end of 1 year, and he then immediately returns the money to the bank, depositing it in a new account. Who has the greater accumulated value at the end of 2 years: Randy or Leonard?

Solution

At the end of two years, Randy has \$112.36:

$$AV_2 = 100(1+0.06)^2 = \$112.36$$

At the end of 1 year, Leonard has \$106:

$$AV_1 = 100(1+0.06) = \$106.00$$

Leonard withdraws the \$106 at the end of 1 year and then deposits the \$106 in a new account, which also earns a compound interest rate of 6% per year. At the end of the second year, Leonard's accumulated value is \$112.36:

$$AV_2 = 106(1+0.06) = \$112.36$$

Leonard and Randy each have \$112.36 at the end of two years. ♦♦

If the type of interest is not specified, the convention is to use compound interest, especially if a period longer than one year is being considered. From now on, a compound interest rate of $x\%$ per year compounded annually will be referred to as an **annual effective interest rate** of $x\%$.

We discuss this in more detail later, but for now we should understand that the effective interest rate is effective over the time period in question, which is one year in the case of an annual effective interest rate.

Example 1.6

Carmen borrows \$1,000 for 90 days at an annual effective interest rate of 8.25%. What will the total payment be in 90 days to repay both the loan principal and interest due on the loan?

Solution

The term of the loan is 3 months, so $t = 3/12 = 0.25$. The total payment is:

$$AV_{0.25} = 1{,}000(1.0825)^{0.25} = \$1{,}020.02$$ ♦♦

Example 1.7

Sam borrows \$20,000. He repays the loan 4 years later with a payment of \$26,709.38. What is the annual effective interest rate on the loan?

Solution

We have $X = 20{,}000$, $t = 4$, and $AV_4 = 26{,}709.38$. Solving for i we have:

$$26{,}709.38 = 20{,}000(1+i)^4$$
$$\Rightarrow 1+i = \left(\frac{26{,}709.38}{20{,}000}\right)^{1/4}$$
$$\Rightarrow i = 7.5\%$$ ♦♦

1.4 *Accumulated value*

We have already learned how to accumulate a single payment into the future. The same can be done for several payments. In later chapters, we'll develop useful notation and formulas to do this, but for now we'll just use the basic principles that we've learned so far.

Compound interest accumulated value factor

Under compound interest, the accumulated value after t years of a deposit of $1 is the compound interest accumulated value factor:

$$AVF_t = (1+i)^t$$

Simple interest accumulated value factor

Under simple interest, the accumulated value after t years of a deposit of $1 is the simple interest accumulated value factor:

$$AVF_t = (1+ti)$$

Let's consider the situation when there is more than one deposit. For example, a deposit of $100 is invested today and another $100 deposit is invested at the end of 5 years. Using an annual effective interest rate of 6%, how much is this investment worth at the end of 10 years?

The initial deposit of $100 is invested for 10 years. The accumulated value at time 10 years is:

$$100(1.06)^{10}$$

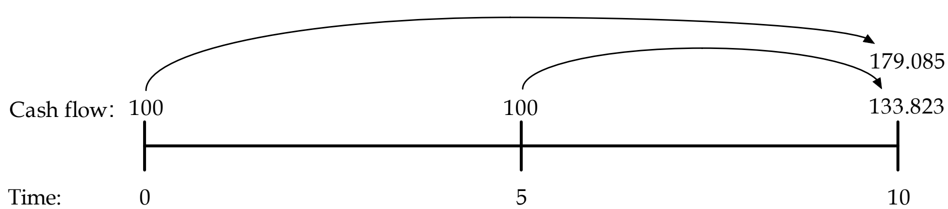
The second deposit is invested for $10-5=5$ years. The accumulated value at time 10 years is:

$$100(1.06)^{5}$$

So, the total accumulated value at time 10 years is:

$$100(1.06)^{10} + 100(1.06)^{5} = 179.085 + 133.823 = \$312.91$$

These cash flows can be illustrated on a timeline.



Timelines such as this are particularly useful when the cash flows are complicated.

Example 1.8

A deposit of $100 is invested today. Another $100 is invested at the end of 5 years. Using an annual simple interest rate of 6%, how much is this investment worth at the end of 10 years?

Solution

Under simple interest, the total investment at time 10 years is worth:

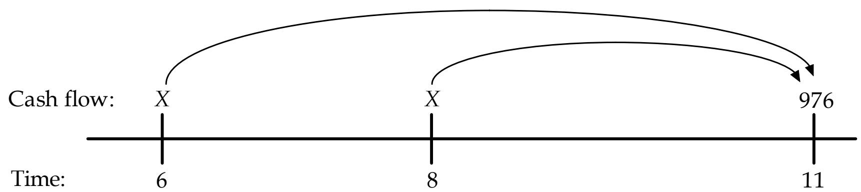
$$100(1+10\times 0.06) + 100(1+5\times 0.06) = \$290.00$$ ♦♦

Example 1.9

A deposit of $\$X$ is invested at time 6 years at an annual effective interest rate of 8%. A second deposit of $\$X$ is invested at time 8 years at the same interest rate. At time 11 years, the accumulated amount of the investment is \$976. Calculate X.

Solution

The first deposit is invested for $11-6=5$ years, and the second deposit is invested for $11-8=3$ years. The timeline is as follows:



The accumulated amount at time 11 years is:

$$X(1.08)^5 + X(1.08)^3 = 976.00$$

Solving this, we have:

$$2.72904X = 976.00$$
$$\Rightarrow \quad X = \$357.63$$

♦♦

Example 1.10

Jim invests \$500 at the beginning of 2002, 2003, and 2004 in a bank account that pays simple interest. At the end of 2004, the accumulated value of the account is \$1,635. Calculate the rate of interest paid by the bank.

Solution

The first deposit is invested for 3 full years; the second deposit for 2 full years; and the final deposit for just one year.

Under simple interest, the total accumulated value at the end of 2004 is:

$$500(1+3i)+500(1+2i)+500(1+i)=500(3+6i)$$

Solving for i, we have:

$$500(3+6i)=1,635$$
$$\Rightarrow i = 4.5\%$$

♦♦

1.5 Present value

Not only can we determine the accumulated value of investments at a future point in time, but we can also find the value now, at time 0, of a payment to be made in the future, taking into account any interest that will be earned during the investment period. This is known as the **present value** of a future payment. The process of allowing for future interest in determining a present value is also known as **discounting** a payment.

The present value of $\$X$ payable in t years is the amount that, if invested now at an annual effective interest rate i, will accumulate to $\$X$ at time t years.

For example, suppose that the annual effective interest rate is 5%, and we need to make a payment of $100 in one year's time. What is the present value of the $100 payable in one year?

Clearly the answer must be less than $100. If we have $100 now, we can invest it at 5% to obtain $105 in one year. So, how much do we need to invest now at 5% in order to have $100 in 1 year? If we denote this unknown quantity as PV, then PV can be found by solving:

$$PV \times 1.05 = 100$$

$$\Rightarrow PV = \frac{100}{1.05} = 95.238095$$

Since $95.24 would accumulate to $100 in 1 year, the present value of $100 in 1 year is $95.24.

Present value

Assuming an annual compound interest rate of i, the present value of a payment of $\$X$ to be made in t years is:

$$PV_t = \frac{X}{(1+i)^t} = X(1+i)^{-t}$$

In interest theory applications, many present values are calculated. Notation has been developed to assist with expressing present values.

General one-year present value factor

The one-year present value factor, which is also known as the one-year discount factor, is:

$$v = \frac{1}{1+i} = (1+i)^{-1}$$

Compound interest present value factor

Under compound interest, the present value of a payment of $1 to be made in t years is the compound interest present value factor:

$$PVF_t = v^t = (1+i)^{-t}$$

Simple interest present value factor

Under simple interest, the present value of a payment of $1 to be made in t years is the simple interest present value factor:

$$PVF_t = (1+ti)^{-1}$$

A payment's present value can be determined as of any point in time. If the valuation date is not specified, we usually assume that we are at time 0 when we calculate present values. However, we can also imagine ourselves to be at a later point in time. So it is also valid to refer to a "present value at time n".

Notice that we have introduced a new variable, v, which is the present value factor for one year. When we study annuities in Chapter 2, we'll find it convenient to write v instead of $(1+i)^{-1}$. The present value factor is simply the inverse of the accumulated value factor. We should also notice that the present value factors for both compound and simple interest are equivalent only when $t = 1$.

Let's derive a formula for i in terms of v. We have:

$$v = \frac{1}{1+i}$$

Rearranging, we have:

$$1+i = \frac{1}{v} \quad \Rightarrow \quad i = \frac{1}{v} - 1 \quad \text{or} \quad i = \frac{1-v}{v}$$

Example 1.11

A payment of \$10 is to be made at time 7 years. Determine the present value of this payment at time 0 and at time 4 years. The annual effective interest rate is 6%.

Solution

The investment is discounted for $7-0=7$ years to determine the present value at time 0:

$$PV_7 = 10v^7 = \frac{10}{(1.06)^7} = \$6.65$$

The present value of an investment at time 4 years of \$10 made at time 7 years is the amount of money that would need to be set aside at time 4 years so that the accumulated value at time 7 is \$10. The investment is discounted for $7-4=3$ years to determine the present value at time 4:

$$PV_3 = 10v^3 = \frac{10}{(1.06)^3} = \$8.40$$ ♦♦

Example 1.12

A payment of \$10 is made at time 7 years. Determine the present value of this payment at time 0 and at time 4 years. The annual simple interest rate is 6%.

Solution

The present value at time 0 under simple interest is:

$$PV_7 = \frac{10}{(1+7\times 0.06)} = \$7.04$$

The present value at time 4 under simple interest is:

$$PV_3 = \frac{10}{(1+3\times 0.06)} = \$8.47$$ ♦♦

1.6 Rate of discount

So far we have considered simple and compound interest rates. Given an annual interest rate i, if an investor deposits or loans \$1 at time 0, a payment of $(1+i)$ is returned at time 1 year. The interest of i is paid at the *end* of the time period.

Another valid approach is to view the interest as being paid at the *beginning* of the time period. When interest is paid at the beginning of the time period, interest is paid in advance, and it is known as **discount**.

Just as with interest, the amount of discount earned is simply the difference between the accumulated account value at the end of the period and the accumulated account value at the beginning of the period.

For any amount of discount earned over a period, we can also calculate an associated discount rate. The discount rate in effect for a one-year period is the amount of discount earned over the year divided by the *ending* accumulated value.

Discount

The amount of discount earned from time t to time $t+s$ is:

$$AV_{t+s} - AV_t$$

where AV_t is the accumulated value at time t.

Discount rate

The annual discount rate d in effect for the year from time t to time $t+1$ is:

$$d = \frac{AV_{t+1} - AV_t}{AV_{t+1}}$$

where t is measured in years.

Let's work through a simple example to explain this. Consider the case of a one-year loan of $1. The lender loans $1 at time 0. The borrower pays discount (*ie* interest) to the lender at time 0, and returns a payment of $1 at time 1 year. The interest is paid at the beginning of the time period.

If interest were payable on the loan at the end of year, the amount of interest payable at that time would be i. But the discount is payable at the beginning of the year, so we must find the present value of the payment of i. Hence, the discount payable at time 0 is:

$$iv = \frac{i}{1+i}$$

The net amount that the borrower receives at time 0 from the loan is the amount of the loan ($1) less the discount ($iv$):

$$\text{Loan amount} - \text{discount} = 1 - iv = 1 - \frac{i}{1+i} = \frac{1}{1+i} = v$$

So, the borrower receives a net payment of v now in exchange for a promise to repay $1 in one year. In other words, the present value at the beginning of the year is v, and the accumulated value at the end of the year is $1. This is consistent with the fact that the present value of $1 payable one year from now is v.

The rate of discount, which is denoted by d, is defined as the amount of the discount (iv) divided by the accumulated value at the end of the year ($1).

Hence, the discount rate is:

$$d = iv$$

We can derive an alternative expression for d:

$$d = iv = \frac{i}{1+i} = 1 - \frac{1}{1+i} = 1 - v$$

This is consistent with the fact that the borrower effectively pays interest of $(1-v)$ at the beginning of the year.

Rate of discount

The annual rate of discount, d, is the amount of interest payable at the start of the year, on a loan of $1 for one year.

$$d = (1 - v) = \frac{i}{1+i} = iv$$

It is very useful to become comfortable converting between these variables. Given that i represents the interest paid at the end of a year, and d represents the discount paid at the start of the year, this relationship makes sense. We can think of d as the present value of a payment of i payable at the end of the year.

We can rearrange the definition to obtain a relationship between the variables i and d:

$$i = \frac{d}{1-d}$$

As we've seen in the previous sections, interest rates can be used to accumulate and discount cash flows. We can also accumulate and discount payments using the rate of discount. (Don't be confused by the fact that discount rates bear the adjective "discount" – in this respect, the terminology can be a little confusing.)

Since $d = (1 - v)$, we have $v = 1 - d$. We already know that v is the one-year discount factor. We also know that the accumulation factor is the inverse of the discount factor. Using this, we can determine the present value and accumulated value factors for compound rates of discount.

Compound rate of discount present value factor

Assuming an annual compound rate of discount of d, the present value of a payment of $1 to be made in t years is the compound rate of discount present value factor:

$$PVF_t = (1-d)^t$$

Compound rate of discount accumulated value factor

Assuming an annual compound rate of discount of d, the accumulated value after t years of a deposit of $1 is the compound rate of discount accumulated value factor:

$$AVF_t = (1-d)^{-t}$$

Example 1.13

An investor would like to have $5,000 at the end of 20 years. The annual compound rate of discount is 5%. How much should the investor deposit today to reach that goal?

Solution

The present value factor for compound discount is $(1-d)^t$.

The investor should set aside:

$$5{,}000(1-d)^t = 5{,}000(1-0.05)^{20} = 5{,}000(0.95)^{20} = \$1{,}792.43$$ ♦♦

Example 1.14

An investor deposits \$1,000 today. The annual compound rate of discount is 6%. What is the accumulated value of the investment at the end of 10 years?

Solution

The accumulated value factor for compound discount is $(1-d)^{-t}$.

The accumulated value at the end of 10 years is:

$$AV_{10} = 1{,}000(1-d)^{-t} = 1{,}000(1-0.06)^{-10} = \$1{,}856.61 \qquad \blacklozenge\blacklozenge$$

We can also have a simple rate of discount instead of a compound rate of discount. Recall that with simple interest, the interest itself does not earn interest.

Simple rate of discount present value factor

Assuming an annual simple rate of discount of d, the present value of a payment of \$1 to be made in t years is the simple rate of discount present value factor:

$$PVF_t = (1-td)$$

Simple rate of discount accumulated value factor

Assuming an annual simple rate of discount of d, the accumulated value after t years of a deposit of \$1 is the simple rate of discount accumulated value factor:

$$AVF_t = (1-td)^{-1}$$

Example 1.15

An investor would like to have \$10,000 at the end of 5 years. The annual simple rate of discount is 3%. How much should the investor deposit today to reach that goal?

Solution

The present value factor for simple discount is $(1-td)$.

The investor should set aside:

$$10{,}000(1-td) = 10{,}000(1-5\times 0.03) = \$8{,}500.00 \qquad \blacklozenge\blacklozenge$$

Example 1.16

An investor deposits \$5,000 today. The annual simple rate of discount is 5%. What is the accumulated value of the investment at the end of 7 months?

Solution

The accumulated value factor for simple discount is $(1-td)^{-1}$. Since we're working with an annual discount rate, the variable t should be expressed in terms of years. So, $t = 7/12$ years.

The accumulated value at the end of 7 months is:

$$AV_{\frac{7}{12}} = 5{,}000(1-td)^{-1} = 5{,}000(1-\frac{7}{12}\times 0.05)^{-1} = \$5{,}150.21 \qquad \blacklozenge\blacklozenge$$

Simple rates of interest and discount are not used very often, and when they are used, it is generally over short periods of time. The next example illustrates why simple discount is suitable only for short periods of time.

Example 1.17

An investor would like to have $5,000 at the end of 20 years. The simple rate of discount is 5%. How much should the investor deposit today to reach that goal?

Solution

The present value factor for simple discount is $(1-td)$. The investor should set aside:

$$PV_{20} = 5,000(1-td) = 5,000(1-20\times 0.05) = 5,000(0) = \$0$$

This seems a little silly, but it is the correct answer if we use a simple rate of discount. The lender would lend $5,000 now. Each year's interest is:

$$\$5,000(0.05) = \$250$$

Therefore, 20 years of interest is $5,000:

$$\$250(20) = \$5,000$$

Subtracting the interest received of $5,000 from the principal of $5,000, the investor's net payment is zero. Since all of the interest is paid up front, the borrower is left with nothing at time 0, but has an obligation to pay $5,000 at time 20 years. This is an excellent deal for the lender, but the reality is that the lender will not find anyone willing to agree to these terms. ♦♦

Simple discount and simple interest are simply unrealistic for use over longer periods of time, so it is usually best to assume that any rate of interest or rate of discount is a compound rate unless otherwise stated.

1.7 Constant force of interest

We have considered the discrete cases where interest is payable at the start or at the end of the year. An annual compound interest rate is the change in the account value over one year, expressed as a percentage of the beginning-of-year value. An annual compound discount rate is the change in the account value over one year, expressed as a percentage of the end-of-year value.

We now consider the case of interest that is **compounded continuously**. A continuously compounded interest rate is called the **force of interest**. The force of interest at time t is denoted δ_t. Non-annual compounding is covered in more detail in Chapter 4.

The force of interest is the *instantaneous* change in the account value, expressed as an annualized percentage of the *current value*.

If the annual effective interest rate is constant, then the force of interest is also constant. When the force of interest is constant, the subscript t is omitted from δ_t since the force of interest does not vary over time. We can derive the value of δ as follows.

In terms of calculus, the force of interest is the derivative of the accumulated value with respect to time expressed as a percentage of the accumulated value at time t:

$$\delta = \frac{AV_t'}{AV_t}$$

where AV_t' is the first derivative of AV_t with respect to t.

Using the standard calculus result:

$$\frac{d}{dx}a^x = a^x \ln a$$

We have:

$$AV_t' = \frac{d}{dt}X(1+i)^t = X(1+i)^t \ln(1+i)$$

Hence we can solve for the force of interest:

$$\delta = \frac{AV_t'}{AV_t} = \frac{X(1+i)^t \ln(1+i)}{X(1+i)^t} = \ln(1+i)$$

Constant force of interest rate

When the force of interest is constant, the force of interest can be expressed in terms of the annual effective interest rate i:

$$\delta = \ln(1+i)$$

For example, if the annual effective interest rate is 5%, then the force of interest is:

$$\delta = \ln(1+i) = \ln(1.05) = 0.048790$$

Let's derive expressions for i and v in terms of δ. Rearranging $\delta = \ln(1+i)$, we have:

$$1+i = e^{\delta} \quad \Rightarrow \quad i = e^{\delta} - 1$$

We also have:

$$v = (1+i)^{-1} = e^{-\delta}$$

We recognize $(1+i)$ as the one-year accumulated value factor and v as the one-year present value factor, so we can now express accumulated values and present values in terms of the constant force of interest.

Constant force of interest accumulated value factor

Assuming a constant force of interest of δ, the accumulated value after t years of a payment of \$1 is the constant force of interest accumulated value factor:

$$AVF_t = e^{\delta t}$$

Constant force of interest present value factor

Assuming a constant force of interest of δ, the present value of a payment of \$1 to be made in t years is the constant force of interest present value factor:

$$PVF_t = e^{-\delta t}$$

Example 1.18

If the constant force of interest is 6%, what is the corresponding annual effective rate of interest?

Solution

The corresponding annual effective rate of interest is calculated as follows:

$$i = e^{\delta} - 1 = e^{0.06} - 1 = 0.06184 \quad \text{or } 6.184\% \qquad \blacklozenge\blacklozenge$$

Example 1.19

Using a constant force of interest of 4.2%, calculate the present value of a payment of \$1,000 to be made in 8 years' time.

Solution

The present value is:

$$1{,}000e^{-8\delta} = 1{,}000e^{-8\times 0.042} = \$714.62 \qquad \blacklozenge\blacklozenge$$

Example 1.20

A deposit of \$500 is invested at time 5 years. The constant force of interest is 6% per year. Determine the accumulated value of the investment at the end of 10 years.

Solution

The money is invested for $10 - 5 = 5$ years.

The accumulated amount is:

$$500e^{5\delta} = 500e^{5\times 0.06} = \$674.93 \qquad \blacklozenge\blacklozenge$$

1.8 *Varying force of interest*

In the previous section, the force of interest was constant over time. The force of interest can vary over time, in which case we write it as δ_t. It is defined as:

$$\delta_t = \frac{AV_t'}{AV_t}$$

Using the standard calculus result:

$$\frac{d}{dx}\ln\left(f(x)\right) = \frac{f'(x)}{f(x)}$$

We can write:

$$\frac{d}{dt}\ln(AV_t) = \frac{AV_t'}{AV_t}$$

Hence we have:

$$\delta_t = \frac{d}{dt}\ln(AV_t)$$

Integrating both sides of the equation from t_1 to t_2, where $t_1 < t_2$:

$$\int_{t_1}^{t_2} \delta_t dt = \int_{t_1}^{t_2} \frac{d}{dt}\ln[AV_t]dt = \ln[AV_{t_2}] - \ln[AV_{t_1}] = \ln\left[\frac{AV_{t_2}}{AV_{t_1}}\right]$$

Hence:

$$\exp\left[\int_{t_1}^{t_2} \delta_t dt\right] = \frac{AV_{t_2}}{AV_{t_1}}$$

Let's consider the meaning of this important result.

The accumulated value at time t_2 divided by the accumulated value at time t_1 is equal to 1 plus the percentage increase in the accumulated value from time t_1 to time t_2.

Hence the accumulated value at time t_2 of $1 invested at time t_1 is given by:

$$\exp\left[\int_{t_1}^{t_2} \delta_t dt\right]$$

Before, we only used one subscript of t to simplify the notation when we have a deposit at time 0 that is accumulated to a later time t. When the deposit is made at a time other than time 0, the following notation is more useful.

Varying force of interest accumulated value factor

The varying force of interest accumulated value factor at time t_2 of an investment of $1 made at time t_1 is:

$$AVF_{t_1,t_2} = \exp\left[\int_{t_1}^{t_2} \delta_t dt\right]$$

Example 1.21

A deposit of $10 is invested at time 2 years. Using a force of interest of $\delta_t = 0.2 - 0.02t$, find the accumulated value of this payment at the end of 5 years.

Solution

The accumulated value is:

$$AV_{2,5} = 10\exp\left[\int_2^5 (0.2 - 0.02t)\ dt\right] = 10\exp\left[\left(0.2t - 0.01t^2\right)\Big|_2^5\right]$$
$$= 10e^{0.75-0.36} = 10e^{0.39} = \$14.77 \qquad \blacklozenge\blacklozenge$$

In a similar manner, we can determine the present value of a payment using a varying force of interest. We recall that the present value is the inverse of the accumulated value.

Varying force of interest present value factor

The varying force of interest present value factor at time t_1 of a payment of $1 to be made at time t_2 is:

$$PVF_{t_1,t_2} = \exp\left[-\int_{t_1}^{t_2} \delta_t\ dt\right]$$

Notice that the only difference between the present value and the accumulated value using a non-constant force of interest is the negative sign in the exponent of the present value equation.

Example 1.22

A deposit of $200 is invested at time 8 years. Using a force of interest of $\delta_t = 0.1 - 0.002t$, find the present value of this payment at the end of 3 years.

Solution

The present value is:

$$PV_{3,8} = 200\exp\left[-\int_3^8 (0.1 - 0.002t)\,dt\right] = 200\exp\left[-\left(0.1t - 0.001t^2\right)\Big|_3^8\right]$$

$$= 200e^{-(0.736-0.291)} = 200e^{-0.445} = \$128.16$$ ♦♦

1.9 Discrete changes in interest rates

The results and techniques used so far in this chapter can be applied to other situations. This section looks at some examples where the interest rate changes at specific points in time. For example, the annual effective interest rate might be $x\%$ for the first m years and then $y\%$ for the next n years.

Example 1.23

A deposit of $100 is invested at time 0. The annual effective interest rate is 5% from time 0 to time 7 years and thereafter is 6%. Calculate the accumulated value of the investment at time 10 years.

Solution

At time 7 years, the accumulated value of the investment is:

$$AV_7 = 100(1.05)^7$$

This amount is then accumulated for a further 3 years at a rate of 6% per year. The accumulated value at time 10 is:

$$AV_{10} = 100(1.05)^7(1.06)^3 = \$167.59$$ ♦♦

Example 1.24

A deposit of $2,500 is invested at time 0. The annual effective rate of interest is 2.5% from time 0 to time 6 years. The annual effective rate of discount is 2.5% from time 6 to time 10 years and the annual force of interest is 2.5% thereafter. Find the accumulated value of the investment at time 13 years.

Solution

The accumulation factor from time 0 to time 6 years is:

$$AVF_{0,6} = 1.025^6$$

The accumulation factor from time 6 to time 10 years is:

$$AVF_{6,10} = (1-0.025)^{-4} = 0.975^{-4}$$

The accumulation factor from time 10 to time 13 years is:

$$AVF_{10,13} = e^{3\times 0.025}$$

The accumulated value of the investment from time 0 to time 13 years is therefore:

$$\begin{aligned} AV_{0,13} &= 2,500 \times AVF_{0,6} \times AVF_{6,10} \times AVF_{10,13} \\ &= 2,500 \times 1.025^{6} \times 0.975^{-4} \times e^{3\times 0.025} \\ &= \$3,458.09 \end{aligned}$$

♦♦

Chapter 1 Practice Questions

Question guide

- Questions 1.1 – 1.10 test material from Sections 1.1 – 1.5.
- Questions 1.11 – 1.16 test material from Sections 1.6 – 1.8.
- Questions 1.17 – 1.20 are from the SOA/CAS Course 2 exam or the IOA/FOA 102 exam.

Question 1.1

$300 is deposited in a bank account, which pays simple interest of 3.5% a year. Calculate the accumulated value of the deposit after 6 years.

Question 1.2

Fund P earns interest at a simple rate of 4% a year. Fund Q earns interest at a simple rate of i% a year. $100 is invested in fund P and $118.50 is invested in fund Q. The accumulated amount in the two funds will be equal after 5 years. Determine the simple rate of interest i.

Question 1.3

$800 is deposited in a bank account, which pays compound interest of 5.25% a year. Calculate the accumulated value of the deposit after 15 years.

Question 1.4

Fund P earns interest at a compound rate of 4% a year. Fund Q earns interest at a compound rate of i% a year. $100 is invested in fund P and $118.50 is invested in fund Q. The accumulated amount in the two funds will be equal after 5 years. Determine the compound rate of interest i.

Question 1.5

Using an annual effective rate of interest of 5%, find the present value of an investment of $5,000 at time 20 years.

Question 1.6

Your great-great-great grandfather set aside $10 on July 1, 1876 in an account paying 5% annual effective interest. Assuming no additional deposits or withdrawals were made, what is the account balance on January 1, 2004? What would the account balance have been if the account had been paying an annual effective interest rate of 10% instead of 5% all of these years?

Question 1.7

Larry has a credit card balance of $10,000. The annual effective interest rate on the credit card is 15%. Larry takes out a home equity loan to pay off his credit card balance. The interest rate on the home equity loan is 5%. Ignoring taxes, how much does this strategy save Larry, assuming he pays off the loan in full 18 months from now?

Question 1.8

Using an annual effective rate of discount of 5% per year, find the accumulated value at time 20 years of an investment of $5,000 at time 0.

Question 1.9

Using a simple rate of discount of 4% per year, find the present value of a payment of $5,000 at time 3 months.

Question 1.10

A $15,000 car loan is repaid with one payment of $18,375.65 after 36 months. What is the annual effective discount rate?

Question 1.11

$500 is paid at time 8 years at a constant force of interest of 10%. Determine the present value of the investment at time 0.

Question 1.12

$1,000 is paid at time 6 years. Find the present value at time 4 years using a force of interest of $\delta_t = 0.05 + 0.002t$.

Question 1.13

Find the accumulated value at time 5 years of $30 that is invested at time 0. Use a force of interest of $\delta_t = 0.02t + 0.01$.

Question 1.14

A payment of $1,000 is due at time 15 years. Between times 0 to 5 years, the annual effective rate of interest is 7%. Between times 5 and 10 years it is 9% and between times 10 and 15 years it is 4%. Calculate the present value of the payment at time 0.

Question 1.15

$100 is invested at time 0. The constant force of interest is 7% from time 0 to time 5 years and 5% from time 5 to time 8 years. Determine the accumulated value of the investment at time 8 years.

Question 1.16

The effective annual rate of discount has been 4% for the last 5 years. Prior to that, it was 5%. A bank account has a balance of $457 today. A single deposit of $*X* was placed in the account 8 years ago. Calculate the value of X.

Question 1.17 **SOA/CAS**

Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits \$100 into his bank account, and Robbie deposits \$50 into his. Each account earns an annual effective discount rate of d. The amount of interest earned in Bruce's account during the 11th year is equal to \$*X*. The amount of interest earned in Robbie's account during the 17th year is also equal to \$*X*. Calculate *X*.

Question 1.18 **SOA/CAS**

Ernie makes deposits of \$100 at time 0, and \$*X* at time 3. The fund grows at a force of interest $\delta_t = 0.01t^2$, $t > 0$. The amount of interest earned from time 3 to time 6 is \$*X*.

Calculate *X*.

Question 1.19 **SOA/CAS**

David can receive one of the following two payment streams:

(i) \$100 at time 0, \$200 at time n, and \$300 at time $2n$
(ii) \$600 at time 10.

At an annual effective interest rate of i, the present values of the two streams are equal.

Given $v^n = 0.75941$, determine i.

Question 1.20 **IOA/FOA**

The force of interest, δ_t is:

$$\delta_t = \begin{cases} 0.04 & 0 < t \le 5 \\ 0.01(t^2 - t) & t > 5 \end{cases}$$

Calculate the present value of \$100 payable at time 10.