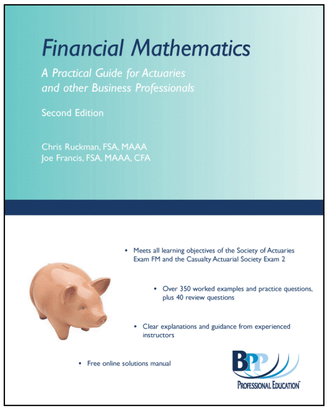

Financial Mathematics
Second Edition

A Practical Guide for Actuaries and other Business Professionals

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Solutions to review questions

Solution 1

Answer: **A**

To determine the amount of interest earned by Dick in the 15th year, we first need to determine the account balance at the end of the 14th year:

$$B_{14} = 500(1+i)^{14}$$

The amount of interest earned by Dick during the 15th year is the annual effective interest rate *i* times the balance at the end of the 14th year.

To determine the amount of interest earned by Bruce during the 7th year, we need to determine the account balance at the end of the 6th year:

$$B_6 = 711.05(1+i)^6$$

The amount of interest earned by Bruce during the 7th year is the annual effective interest rate *i* times the balance at the end of the 6th year.

We are given that the amount of interest earned by Dick during his 15th year and the amount of interest earned by Bruce during his 7th year are equal. We set up the equation of value and solve for *i*:

$$\begin{aligned} 500(1+i)^{14} i &= 711.05(1+i)^6 i \\ \frac{(1+i)^{14}}{(1+i)^6} &= \frac{711.05}{500} \\ (1+i)^8 &= 1.4221 \\ i &= 0.045 \end{aligned}$$

Solution 2

Answer: **B**

Sheryl's equation of value is:

$$500+1{,}000v^3+1{,}500v^6=2{,}276.90$$

We let $x=v^3$, and we can solve for x using the quadratic equation:

$$1{,}500x^2+1{,}000x+500=2{,}276.90$$
$$1{,}500x^2+1{,}000x-1{,}776.90=0$$
$$x=\frac{-1{,}000\pm\sqrt{1{,}000^2-4(1{,}500)(-1{,}776.90)}}{2(1{,}500)}$$
$$x=0.80496 \quad \text{or} \quad x=-1.47163$$

We discard the negative solution, and we can solve for the annual effective interest rate:

$$(1+i)^{-3}=0.80496$$
$$1+i=(0.80496)^{-1/3}$$
$$i=0.075$$

Solution 3

Answer: **E**

The \$500 deposit accumulates for n years. The \$750 deposit accumulates for $n-5$ years. The \$1,500 deposit accumulates for $n-10$ years. We set up the equation of value for the accumulated value after n years and solve for n:

$$500(1.12)^n+750(1.12)^{n-5}+1{,}500(1.12)^{n-10}+12{,}131.33=24{,}262.66$$
$$500(1.12)^n+750(1.12)^n(1.12)^{-5}+1{,}500(1.12)^n(1.12)^{-10}=12{,}131.33$$
$$1{,}408.53000(1.12)^n=12{,}131.33$$
$$n\ln(1.12)=\ln 8.61276$$
$$n=\frac{\ln(8.61276)}{\ln(1.12)}$$
$$n=19.0$$

Solution 4

Answer: **B**

The accumulated value at time 4 years is:

$$AV_4=\exp\left[\int_0^4 0.025t\,dt\right]=\exp\left[\left.\frac{0.025t^2}{2}\right|_0^4\right]=\exp\left[\left.0.0125t^2\right|_0^4\right]=\exp\left[0.0125(16-0)\right]=\exp\left[0.20\right]=1.22140$$

The accumulated value at time 5 years is:

$$AV_5=1.22140e^{0.10\times 1}=1.34986$$

The annual effective interest rate earned over the first five year period is:

$$i = 1.34986^{1/5} - 1 = 0.0618$$

Solution 5

Answer: **D**

Answer choice (D) is not a correct expression for $a_{\overline{n}|}$ because the factor that should be applied to the continuously paid annuity present value is delta over *i*, not *i* over delta:

$$a_{\overline{n}|} = (\delta / i)\bar{a}_{\overline{n}|}$$

Solution 6

Answer: **D**

First we need to determine the annual effective interest rate from the present value of a perpetuity-immediate that pays $25,000 at the end of each year forever:

$$PV = 416,666.67 = \frac{25,000}{i}$$
$$i = 0.06$$

Mike receives 10 annual payments of $25,000 from time 1 year to time 10 years. The present value of Mike's payments is:

$$25,000a_{\overline{10}|} = 25,000\frac{1-(1.06)^{-10}}{0.06} = 184,002.18$$

Ann receives 25 annual payments of $25,000 from time 11 years to time 35 years. The annuity-immediate present value factor is valued one year before the first payment (*ie* time 10), so we need to discount the present value factor for another 10 years to value the cash flows at time 0. The present value of Ann's payments is:

$$25,000v^{10}a_{\overline{25}|} = 25,000(1.06)^{-10}\frac{1-(1.06)^{-25}}{0.06} = 178,453.98$$

Liz receives an infinite number of annual payments of $25,000 starting at time 36 years and continuing forever. The annuity-immediate present value factor is valued one year before the first payment (*ie* time 30), so we need to discount the present value for another 35 years to value the cash flows at time 0. The present value of Liz's payments is:

$$\frac{25,000(1.06)^{-35}}{0.06} = 54,210.51$$

So, Mike's present value is greater than Ann's present value, which is greater than Liz's present value.

Solution 7

Answer: **D**

Cindy deposits $*X* at the beginning of each year, from time 0 to time 9 years. The accumulated value of this payment series at time 10 years is:

$$X\ddot{s}_{\overline{10}|4\%}$$

The present value at time 10 years of 25 level annual payments of \$50,000 which are paid from time 10 years to time 34 years is:

$$25,000\ddot{a}_{\overline{25}|4\%}$$

Since these are both time 10 values, we can equate the two expressions and solve for X:

$$X\left[\frac{1.04^{10}-1}{0.04/1.04}\right]=50,000\left[\frac{1-1.04^{-25}}{0.04/1.04}\right]$$

$$X=\frac{50,000(16.24696)}{12.48635}\quad .$$

$$X=65,058.89$$

Solution 8

Answer: **C**

The present value of the perpetuity-due is used to determine the annual effective discount rate:

$$PV=\ddot{a}_{\overline{\infty}|}$$

$$1,250=\frac{100}{d}$$

$$d=\frac{100}{1,250}=0.08$$

The annual effective interest rate is:

$$i=\frac{0.08}{1-0.08}=0.086957$$

The present value of the annuity-immediate is the used to determine n:

$$PV=152.30a_{\overline{n}|}$$

$$1,250=152.30\left[\frac{1-1.086957^{-n}}{0.086957}\right]$$

$$0.713694=1-1.086957^{-n}$$

$$n=\frac{-\ln(0.286306)}{\ln(1.086957)}$$

$$n=15.0$$

Solution 9

Answer: **B**

Wally's present value can be used to determine the annual effective discount rate, the annual effective interest rate, and the force of interest:

$$PV=10(I\ddot{a})_{\overline{\infty}|}$$

$$2,684.56=\frac{10}{d^2}$$

$$d=0.061033$$

$$i=\frac{0.061033}{1-0.061033}=0.065$$

$$\delta=\ln(1.065)=0.062975$$

Ron's present value is then:

$$X = 9(I\bar{a})_{\overline{\infty|}} = \frac{9}{\delta d} = \frac{9}{(0.062975)(0.061033)} = 2,341.60$$

Solution 10

Answer: **E**

The payments start at $100 paid continuously during the first year, and they decrease by $5 per year until the last year when $55 is paid continuously over the tenth year. We can split the payment stream into two parts: a level $50 per year continuously paid annuity and a decreasing continuously paid annuity with $50 paid during the first year, $45 paid during the second year, and so on, down to $5 paid during the tenth year. The accumulated value of the payment stream at time 10 years is then:

$$AV_{10} = 50\bar{s}_{\overline{10|}} + 5(D\bar{s})_{\overline{10|}}$$

Calculating the required values, we have:

$$\delta = \ln(1.045) = 0.044017$$

$$s_{\overline{10|}} = \frac{(1.045)^{10} - 1}{0.045} = 12.288209$$

$$\bar{s}_{\overline{10|}} = \frac{(1.045)^{10} - 1}{0.044017} = 12.562666$$

$$(D\bar{s})_{\overline{10|}} = \frac{10(1.045)^{10} - 12.288209}{0.044017} = 73.641849$$

The accumulated value at time 10 years is:

$$AV_{10} = 50(12.562666) + 5(73.641849) = 996.342538$$

The accumulated value at time 15 years is:

$$AV_{15} = 996.342538(1.045)^5 = 1,241.624$$

Solution 11

Answer: **D**

The payments start at $50 at time 0 and increase by $5 per year until the 25th payment of $170 occurs at time 24 years. We can split the payment stream into two parts: a 25-year level $50 annuity-due and an increasing annuity-immediate in which the first payment of $5 is made at time 1 year, and with the payments increasing by $5 per year until the 24th payment of $120 occurs at time 24 years. The present value of the payment stream is:

$$PV = 50\ddot{a}_{\overline{25|}} + 5(Ia)_{\overline{24|}}$$

Calculating the required values, we have:

$$\ddot{a}_{\overline{25|}} = \frac{1 - 1.04^{-25}}{0.04/1.04} = 16.24696$$

$$\ddot{a}_{\overline{24|}} = \frac{1 - 1.04^{-24}}{0.04/1.04} = 15.85684$$

$$(Ia)_{\overline{24|}} = \frac{15.85684 - 24(1.04)^{-24}}{0.04} = 162.34816$$

The present value is:

$$50(16.24696)+5(162.34816)=1,624.09$$

Solution 12

Answer: **B**

Dan would like to have \$1,000,000 in his account in exactly 30 years. The present value of \$1,000,000 is:

$$1,000,000(1.04)^{-30}=308,318.668$$

We can use the present value formula for the compound increasing annuity-due even though the growth rate is greater than the interest rate. The net interest rate is:

$$j=\frac{0.04-0.05}{1.05}=-0.009524$$

The present value of the 30 payments is:

$$X\ddot{a}_{\overline{30}|-0.9524\%}=X\frac{1-[1+(-0.009524)]^{-30}}{-0.009525/[1+(-0.009524)]}=34.583694X$$

Equating these two present values, we solve for X:

$$X=\frac{308,318.668}{34.583694}=8,915.15$$

Alternatively, we can use the finite geometric series to determine X:

$$X+X\frac{1.05}{1.04}+X\frac{1.05^2}{1.04^2}+\cdots+X\frac{1.05^{29}}{1.04^{29}}=308,318.668$$

$$X\left[\frac{1-(1.05/1.04)^{30}}{1-(1.05/1.04)}\right]=308,318.668$$

$$X=\frac{308,318.668}{34.583694}$$

$$X=8,915.15$$

Solution 13

Answer: **E**

By relating the nominal interest rate convertible quarterly with the nominal discount rate convertible semiannually, we can solve for the nominal interest rate convertible quarterly:

$$\left(1+\frac{i^{(4)}}{4}\right)^4=\left(1-\frac{d^{(2)}}{2}\right)^{-2}$$

$$\left(1+\frac{i^{(4)}}{4}\right)^4=\left(1-\frac{0.15}{2}\right)^{-2}$$

$$\left(1+\frac{i^{(4)}}{4}\right)^4=(0.925)^{-2}$$

$$\left(1+\frac{i^{(4)}}{4}\right)^4=1.168736$$

$$1+\frac{i^{(4)}}{4}=1.039750$$

$$\frac{i^{(4)}}{4}=0.039750$$

$$i^{(4)}=0.1590$$

Solution 14

Answer: **C**

Let i be the annual effective interest rate. The present value of the annuity is:

$$PV=15\left[v^5+v^{10}+v^5+\cdots+v^{50}\right]$$

Simplifying, and using the finite geometric series, we have:

$$\begin{aligned}PV&=15v^5\left[1+v^5+v^{10}+\cdots+v^{45}\right]\\&=15v^5\left[\frac{1-v^{50}}{1-v^5}\right]\\&=\frac{15v^5\left[1-v^{50}\right]}{v^5\left[v^{-5}-1\right]}\\&=15\frac{\left[1-v^{50}\right]/i}{\left[(1+i)^5-1\right]/i}\\&=15a_{\overline{50|}}/s_{\overline{5|}}\end{aligned}$$

Solution 15

Answer: **D**

The quarterly increasing annuity-immediate accumulated value factor expects a payment of 1/4 at the end of each quarter during the first year, 2/4 at the end of each quarter during the second year, and so on, up to a payment of 15/4 at the end of each quarter during the 15th year. In this case, we have payments of 5 at the end of each quarter during the first year, 10 at the end of each quarter during the second year, and so on, up to a payment of 75 at the end of each quarter during the 15th year. So we need to apply a factor of $5 \times 4 = 20$ to this accumulated value factor to match the cash flow pattern. The accumulated value at time 15 years is:

$$20(Is)^{(4)}_{\overline{15}|}$$

Calculating the required values, we have:

$$d = \frac{0.08}{1.08} = 0.074074$$

$$i^{(4)} = 4\left[(1.08)^{1/4} - 1\right] = 0.077706$$

$$\ddot{s}_{\overline{15}|} = \frac{1.08^{15} - 1}{0.074074} = 29.324283$$

$$(Is)^{(4)}_{\overline{15}|} = \frac{29.324283 - 15}{0.077706} = 184.33903$$

Hence, the accumulated value at time 15 years is:

$$20(184.33903) = 3,686.78$$

Solution 16

Answer: **A**

The \$500 deposit accumulates for 4 two-year year periods to time 8 years, and the \$1,000 deposit accumulates for 2 two-year year periods to time 8 years. In this case, $j = i^{(1/2)}$. The accumulated value at time 8 years is:

$$500\left(1 + \frac{j}{1/2}\right)^4 + 1,000\left(1 + \frac{j}{1/2}\right)^2 = 2,041.16$$

We let $x = \left(1 + \frac{j}{1/2}\right)^2$, and we can use the quadratic equation to solve for x:

$$500x^2 + 1,000x - 2,041.16 = 0$$

$$x = \frac{-1,000 \pm \sqrt{1,000^2 - 4(500)(-2,041.16)}}{2(500)}$$

$$x = 1.25440 \quad \text{or} \quad x = -3.25440$$

We discard the negative solution, and we solve for j:

$$\left(1 + \frac{j}{1/2}\right)^2 = 1.25440$$

$$1 + \frac{j}{1/2} = 1.12$$

$$j = 0.06$$

Solution 17

Answer: **C**

Let's work in quarters. Starting with the nominal interest rate convertible monthly, we determine the quarterly effective interest rate:

$$i^{(12)} = 0.09$$
$$\frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$
$$\frac{i^{(4)}}{4} = (1.0075)^3 - 1 = 0.022669$$

There are 5 years of quarterly payments, so there are 20 quarterly payments, starting at \$500 at time 0 and decreasing by \$10 each quarter until the last payment of \$310 at time 19 quarters. We can split this payment stream into two parts: a level \$300 quarterly annuity-due and a quarterly decreasing annuity due, starting with a payment of \$200 at time 0 and decreasing by \$10 per quarter until the last payment of \$10 at time 19 quarters. The present value of this payment series is:

$$PV = 300\ddot{a}_{\overline{20}|2.2669\%} + 10(D\ddot{a})_{\overline{20}|2.2669\%}$$

Calculating the required values, we have:

$$\ddot{a}_{\overline{20}|2.2669\%} = \frac{1 - 1.022669^{-20}}{0.022669 / 1.022669} = 16.29926$$
$$a_{\overline{20}|2.2669} = \frac{16.29926}{1.022669} = 15.93796$$
$$\frac{d^{(4)}}{4} = \frac{0.022669}{1.022669} = 0.022167$$
$$(D\ddot{a})_{\overline{20}|} = \frac{20 - 15.93796}{0.022167} = 183.24999$$

The present value is then:

$$300(16.29926) + 10(183.24999) = 6{,}722.28$$

Solution 18

Answer: **A**

The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

To determine the amount of interest due in the tenth monthly payment, we first need to determine the loan balance at the end of the 9th month. Using the retrospective method, we have:

$$\begin{aligned} B_9 &= 4{,}000(1.01)^9 - 100 s_{\overline{9}|1\%} \\ &= 4{,}000(1.01)^9 - 100\frac{1.01^9 - 1}{0.01} \\ &= 3{,}437.88836 \end{aligned}$$

The amount of interest due in the tenth payment is:

$$I_{10} = B_9 \frac{i^{(12)}}{12} = 3,437.88836(0.01) = 34.38$$

Solution 19

Answer: **B**

We are given that the investor makes level deposits into a bank account at the end of each year for 10 years and that it earns interest at the end of every year at an annual effective interest rate of 15%. The interest earns interest at an annual effective rate of 11%.

At the end of the first year, the first investment of \$250 is deposited. At the end of the second year, the first investment of \$250 has earned 0.15×250 in interest, which is reinvested at 11% for 8 years to time 10 years. Also at the end of the second year, a second investment of \$250 is deposited. At the end of the third year, two payments of \$250 have earned 0.15×250 in interest, which is reinvested at 11% for 7 years until time 10 years. At the end of the fourth year, three payments of \$250 have earned 0.15×250 in interest, which is reinvested at 11% for 6 years until time 10 years. That pattern continues, until at the end of ten years, nine payments of 250 have earned 0.15×250 in interest, but since this occurs at time 10, there is no time for this to be reinvested at 11% until time 10 years.

In total, there are 10 annual payments of 250 into the fund. The total investment over 10 years is 10×250. We can set up the equation for this series of investments valued at time 10 years as:

$$10 \times 250 + 0.15(250)(1.11)^8 + 2 \times 0.15(250)(1.11)^7 + 3 \times 0.15(250)(1.11)^6 + \cdots + 9 \times 0.15(250)(1.11)^0$$

This can be reduced to:

$$2,500 + 0.15(250)\left[1(1.11)^8 + 2(1.11)^7 + 3(1.11)^6 + \cdots + 9(1.11)^0\right]$$

The part in the brackets is the accumulated value of an increasing annuity-immediate, or $(Is)_{\overline{9}|11\%}$. Determining the required values, we have:

$$\ddot{s}_{\overline{9}|11\%} = \frac{(1.11)^9 - 1}{0.11/1.11} = 15.72201$$

$$(Is)_{\overline{9}|11\%} = \frac{15.72201 - 9}{0.11} = 61.10917$$

The present value is:

$$2,500 + 37.5(61.10917) = 4,791.59$$

Solution 20

Answer: **E**

The \$200 deposit accumulates for 12 months, the \$100 withdrawal accumulates for 9 months, the \$400 deposit accumulates for 4 months, and the \$300 withdrawal accumulates for 1 month. To determine the dollar-weighted interest rate (*ie* IRR), we set up the equation of value:

$$200(1+i)^{12/12} - 100(1+i)^{9/12} + 400(1+i)^{4/12} - 300(1+i)^{1/12} = 227.79$$

Since the time period is one year, we can use the simple interest approximation to determine the approximate dollar-weighted interest rate:

$$\begin{aligned} 200(1+1i)-100(1+0.75i)+400(1+0.33333i)-300(1+0.08333i) &= 227.79 \\ 233.33333i &= 27.79 \\ i &= 11.91\% \end{aligned}$$

To determine the time-weighted interest rate, we set up the growth factor equation and solve for i:

$$\begin{aligned} 1+i &= \left(\frac{203.89}{200}\right)\left(\frac{108.10}{203.89-100}\right)\left(\frac{526.16}{108.10+400}\right)\left(\frac{227.79}{526.16-300}\right) \\ 1+i &= 1.10638 \\ i &= 10.64\% \end{aligned}$$

The approximate dollar-weighted rate minus the time-weighted rate is $11.91\%-10.64\%=1.27\%$.

Using a spreadsheet to determine the actual dollar-weighted interest produces a value of 12.05% instead of 11.91%, so the actual dollar-weighted rate minus the time-weighted rate is 1.41%. Nevertheless, since the time period is one year, the approximation is accurate enough to provide the correct answer.

Solution 21

Answer: **B**

The monthly effective interest rate is:

$$\frac{i^{(12)}}{12}=(1.06)^{1/12}-1=0.004868$$

At month 24, there are 24 months remaining on the original loan. According to the prospective method, the balance at this time (before the additional payment of $500) is:

$$B_{24}=300a_{\overline{24}|}=300\frac{1-(1.004868)^{-24}}{0.004868}=6,779.81$$

An additional payment of $500 is made at this time, so the balance to be repaid over a new 3-year loan period is $6,529.81. The 36-month annuity-immediate present value factor is:

$$a_{\overline{36}|}=\frac{1-(1.004868)^{-36}}{0.004868}=32.94896$$

The new monthly payment is:

$$P=\frac{6,279.81}{32.94896}=190.59$$

Solution 22

Answer: **E**

The interest due on the loan each year is $0.06\times 15,000=\$900$. Lynn's service payment to the lender each year is $1,000, so we need to use the general equation of value for the sinking fund since the service payment does not equal the interest due on the loan. The general sinking fund equation of value is:

$$15,000(1.06)^5=1,000s_{\overline{5}|6\%}+SFP\times s_{\overline{5}|4\%}$$

Calculating the required values, we have:

$$s_{\overline{5}|6\%} = \frac{(1.06)^5 - 1}{0.06} = 5.63709$$

$$s_{\overline{5}|4\%} = \frac{(1.04)^5 - 1}{0.06} = 5.41632$$

We rearrange the sinking fund equation of value and solve for the sinking fund payment:

$$SFP = \frac{15,000(1.33823) - 1,000(5.63709)}{5.41632} = 2,665.33$$

The total annual payment on the loan is the service payment plus the sinking fund payment:

$$1,000 + 2,665.33 = 3,665.33$$

Solution 23

Answer: **D**

Interestingly, we don't need to determine n to answer this question. The semiannual effective yield, the semiannual effective coupon rate, and the semiannual coupon amount are:

$$i = (1.1025)^{1/2} - 1 = 0.05$$

$$r = \frac{0.08}{2} = 0.04$$

$$Fr = 1,000(0.04) = 40$$

The retrospective formula to determine the book value at time t accumulates the initial price to time t and subtracts the accumulated value of the coupons up to time t. We can use this formula to determine the initial price of the bond. The book value at time 8 semiannual periods is \$900:

$$BV_8 = 900 = P(1.05)^8 - 40s_{\overline{8}|5\%}$$

$$P = \frac{900 + 40\left(\dfrac{(1.05)^8 - 1}{0.05}\right)}{(1.05)^8} = 867.68394$$

Once we know the initial bond price, we can use the retrospective formula once again to determine the book value at time 16 semiannual periods:

$$\begin{aligned} BV_{16} &= (867.68394)(1.05)^{16} - 40\left(\frac{(1.05)^{16} - 1}{0.05}\right) \\ &= 947.75 \end{aligned}$$

Solution 24

Answer: **A**

The semiannual effective coupon rate for the coupon bond is $0.08/2 = 0.04$ and the semiannual coupon is $0.04 \times 1,000 = 40$. There are 40 coupon periods, and the bond matures for \$1,050. We let the semiannual effective yield be i. The formula to determine the price of the bond is:

$$P = 40a_{\overline{40}|i} + 1,050(1+i)^{-40}$$

The BA-35 calculator quickly determines the semiannual effective yield. We input 40 [PMT], 835.51 [PV], 1,050 [FV], 40 [N], and then [CPT] [%i] and the semiannual effective yield is 5.0%.

This is the same yield earned by the zero-coupon bond. The zero-coupon bond has 20 semiannual periods and it matures for $1,000. The price of this bond is:

$$P = \frac{1{,}000}{(1.05)^{20}} = 376.89$$

Solution 25

Answer: **E**

The bond has 5 semiannual periods until maturity. We can rearrange the bond price formula in terms of the difference between the coupon rate and the yield rate, which we are given, once we recognize that we can rewrite the annuity-immediate present value factor formula as $v^n = 1 - ia_{\overline{n}|}$:

$$\begin{aligned} P_t &= Fra_{\overline{n-t}|} + C(1 - ia_{\overline{n-t}|}) \\ &= C + (Fr - Ci)a_{\overline{n-t}|} \end{aligned}$$

In this case, the face amount equals the redemption amount, $r - i = 0.015$ and $n - t = 5$. We use this formula to solve for the annuity present value factor:

$$P_t = 106.87 = 100 + 100(0.015)a_{\overline{5}|}$$

$$a_{\overline{5}|} = \frac{106.87 - 100}{1.5} = 4.58$$

Using the BA-35 calculator, we can quickly determine the semiannual effective yield i. We input 1 [PMT], 4.58 [PV], 5 [N], and then [CPT][%i], and we get $i = 0.03$. So the semiannual effective coupon rate is $r = 0.03 + 0.015 = 0.045$. The annual coupon rate convertible semiannually is:

$$x = 2 \times 0.045 = 0.09$$

Solution 26

Answer: **D**

In order to be sure of receiving her desired yield, the investor must pay no more than the lowest of the prices, so the investor pays $1,101.52.

The investor pays $1,101.52 and then receives coupons of $70 per year for 15 years at which time the investor also receives $1,050. The internal rate of return is easily determined with a calculator. Using the BA-35 calculator, we input 15 [N], 1,101.52 [PV], 70 [PMT], 1,050 [FV], and then [CPT] [%i], and the result is 6.16%.

Solution 27

Answer: **C**

The margin deposit is 75% of the sale price of the stock:

$$\text{Margin deposit} = (0.75)80 = 60$$

The profit on the short sale is:

$$\begin{aligned} \text{Short sale profit} &= (\#\ \text{shares})(\text{sale price} - \text{purchase price}) + \text{margin interest} - \text{dividends} \\ &= (1)(80 - X) + 5 - 3 = 82 - X \end{aligned}$$

The formula for the short sale yield can now be used to solve for X :

$$\text{Short sale yield} = \frac{\text{short sale profit}}{\text{margin deposit}}$$

$$0.22 = \frac{82 - X}{60} \quad \Rightarrow \quad X = 68.80$$

Solution 28

Answer: **E**

The constant dividend growth stock formula is:

$$PV \text{ stock} = \frac{div_1}{r - g}$$

Substituting the values from the problem allows us to solve for the annual effective interest rate:

$$58 = \frac{5}{i - 0.025} \quad \Rightarrow \quad i = 11.12\%$$

Solution 29

Answer: **D**

The accumulation of the discount in the t-th year is the negative of the premium amortization amount:

$$\text{Accumulation of discount} = -PA_t$$

The premium amortization amount is:

$$PA_t = C(g - i)v^{n-t+1}$$

$$PA_{20} = 100(0.10 - 0.15)\left(\frac{1}{1.15}\right)^{20-20+1} = -4.3478$$

Therefore, the accumulation of discount in the 20th year is \$4.3478.

Solution 30

Answer: **E**

The semiannual effective yield is:

$$1.04^{0.5} - 1 = 0.0198039$$

The price of the bond is:

$$P = Fra_{\overline{n|}i} + Cv_i^n = 35a_{\overline{44|}0.0198039} + \frac{1{,}000}{1.04^{22}} = 1{,}443.55$$

The premium is the purchase price minus the redemption value:

$$\text{Premium} = P - C = 1{,}443.55 - 1{,}000.00 = 443.55$$

Solution 31

Answer: **E**

Recall that Macaulay duration can be viewed as the weighted average of the times that the cash flows occur, where the weights are the present value of the cash flows.

The first bond has the same yield as the second bond, but the first bond's coupons are higher. The higher coupons place relatively more weight on the earlier values of t, so $X < Y$.

The first bond and the third bond have the same coupons, but the third bond has a higher yield. The higher yield on the third bond causes the weights on its later cash flows to be relatively less, so $X > Z$.

Solution 32

Answer: **C**

The price of the bond is:

$$P = \frac{50}{1.1} + \frac{50}{1.1^2} + \frac{50}{1.1^3} + \frac{1{,}050}{1.1^4} = 841.50673$$

The numerator of the Macaulay duration formula is:

$$\sum \frac{tCF_t}{(1+y)^t} = \frac{50(0.5)}{1.1} + \frac{50(1.0)}{1.1^2} + \frac{50(1.5)}{1.1^3} + \frac{1{,}050(2.0)}{1.1^4} = 1{,}554.72645$$

The bond's Macaulay duration is:

$$MacD = \frac{1{,}554.72645}{841.50673} = 1.848$$

Solution 33

Answer: **A**

The duration of a portfolio can be calculated as the weighted average of the bonds' durations, using the market values of the bonds as the weights. The duration of the portfolio is:

$$\frac{1{,}182.56(18.71) + 896.20(11.34) + 556.84(12.00)}{1{,}182.56 + 896.20 + 556.84} = 14.79$$

Note that the Macaulay duration of bond C is 12.0 because the Macaulay duration of a zero-coupon bond is equal to its maturity.

Solution 34

Answer: **B**

First, we note that we do not need the 3-year bond since the final liability cash flow occurs at time 2 years.

Let's assume that each bond has par value of \$100.

The two-year bond has a cash flow of \$103 at time 2 years. We can determine the number two-year bonds needed to provide an asset cash flow of \$100,000 in two years:

$$\frac{100{,}000}{103} = 970.87379$$

The two-year bond also produces a cash flow at time 1 year of $970.8739(3) = 2,912.621359$. So the net cash flow required of the one-year bond is:

$$100,000 - 2,912.621359 = 97,087.37864$$

Since the one-year bond has a cash flow of \$108 at time 1 year, the number of one-year bonds needed to provide an asset cash flow of \$97,087.37864 in one year is:

$$\frac{97,087.37864}{108} = 898.95721$$

The prices (*ie* cost) of the bonds are:

$$P_{2-yr} = \frac{3}{1.06} + \frac{103}{1.06^2} = 94.49982$$
$$P_{1-yr} = \frac{108}{1.06} = 101.88679$$

The cost to establish this portfolio is:

$$970.87379(94.49982) + 898.95721(101.88679) = 183,339.27$$

A shortcut approach to determining the answer can be used when we recognize that both bonds have the same yield. Since both bonds have a yield of 6% and their combined cash flows need to add up to \$100,000 at the end of one year and \$100,000 at the end of two years, we can quickly find the total cost of the bond portfolio as:

$$\frac{100,000}{1.06} + \frac{100,000}{1.06^2} = 183,339.27$$

Solution 35

Answer: **B**

The present value of the liability cash flow must equal the present value of the assets for the position to be immunized. Therefore, the present value of the asset portfolio must be:

$$\frac{10,000}{1.1^3} = 7,513.148$$

The Macaulay duration of the liability is 3. The Macaulay duration of the 2-year bond is 2 since it is a zero-coupon bond. The Macaulay duration of the 5-year bond is:

$$MacD = \ddot{a}^{(1)}_{\overline{5}|10\%} = \frac{1-(1.10)^{-5}}{0.10/1.10} = 4.16987$$

If $a\%$ of the \$7,513.148 is invested in the 5-year bond, the we can solve for the value of $a\%$ that satisfies the second immunization condition that the durations should be equal:

$$(1-a\%)2 + (a\%)4.16987 = 3$$
$$a\% = 46.08581\%$$

So the amount invested in the 5-year bond is:

$$0.4608581(7,513.148) = 3,462.49488$$

Since the coupon rate of the 5-year bond is equal to its yield, we know that it must be priced at par. Therefore, the number of 5-year bonds purchased is:

$$\frac{3,462.49488}{1,000} = 3.46$$

Solution 36

Answer: **D**

The yield rates on zero-coupon bonds are spot rates, so we can use the following formula to find the one-year forward rate (f_1):

$$f_{t-1} = \frac{(1+s_t)^t}{(1+s_{t-1})^{t-1}} - 1$$

$$f_1 = \frac{(1+s_2)^2}{(1+s_1)} - 1 = \frac{(1.09)^2}{1.08} - 1 = 0.10$$

Solution 37

Answer: **A**

The par values of the bonds will not affect the calculated value of the 10-year spot rate, so for convenience we assume that both bonds have par values of $1,000.

The price of the 7% coupon bond is:

$$70a_{\overline{10}|0.08} + \frac{1{,}000}{1.08^{10}} = 932.89919$$

The price of the 8% coupon bond is:

$$80a_{\overline{10}|0.083} + \frac{1{,}000}{1.083^{10}} = 980.13933$$

If we calculate the bond prices using the spot rates, we have the following two equations:

$$932.89919 = 70\left(\frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots \frac{1}{(1+s_{10})^{10}}\right) + \frac{1{,}000}{(1+s_{10})^{10}}$$

$$980.13933 = 80\left(\frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots \frac{1}{(1+s_{10})^{10}}\right) + \frac{1{,}000}{(1+s_{10})^{10}}$$

Subtracting the first equation from the second, we have:

$$47.24014 = 10\left(\frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots \frac{1}{(1+s_{10})^{10}}\right)$$

$$4.724014 = \left(\frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots \frac{1}{(1+s_{10})^{10}}\right)$$

Substituting 4.724014 into the formula for the price of the 7% bond allows us to find the 10-year spot rate:

$$932.89919 = 70\left(\frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots \frac{1}{(1+s_{10})^{10}}\right) + \frac{1{,}000}{(1+s_{10})^{10}}$$

$$932.89919 = 70(4.724014) + \frac{1{,}000}{(1+s_{10})^{10}}$$

$$602.21821 = \frac{1{,}000}{(1+s_{10})^{10}}$$

$$s_{10} = 5.20\%$$

Solution 38

Answer: **E**

The 2-year forward rate is:

$$f_2 = \frac{(1+s_3)^3}{(1+s_2)^2} - 1 = \frac{1.23^3}{1.20^2} - 1 = 0.29227$$

Since the \$500 investment earns the 2-year forward rate over the interval from time 2 years to time 3 years, the investment grows to:

$$500(1.29227) = 646.134$$

Solution 39

Answer: **D**

Each of the cash flows is discounted at the appropriate spot rate:

$$P = \sum_{t>0} \frac{CF_t}{\left(1+s_t\right)^t} = \frac{150}{1.05} + \frac{150}{1.10^2} + \frac{150}{1.13^3} + \frac{1,150}{1.15^4} = 1,028.30$$

Solution 40

Answer: **B**

The price of the one-year bond can be used to find the one-year spot rate:

$$\frac{1,060}{(1+s_1)} = 990.65 \qquad \Rightarrow \qquad s_1 = 7.000\%$$

The price of the two-year bond can now be used to find the two-year spot rate:

$$\frac{40}{1.07000} + \frac{1,040}{(1+s_2)^2} = 929.02 \qquad \Rightarrow \qquad s_2 = 8.000\%$$

The price of the three-year bond can now be used to find the three-year rate:

$$\frac{230}{1.07000} + \frac{230}{1.08000^2} + \frac{1,230}{(1+s_3)^3} = 1,444.87 \qquad \Rightarrow \qquad s_3 = 6.000\%$$