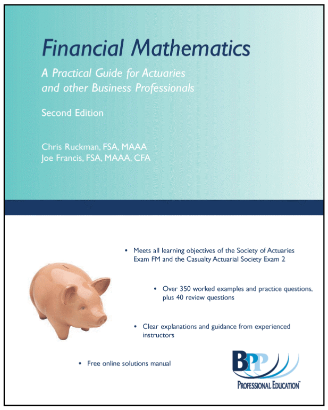

Financial Mathematics
Second Edition

A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 7

Solution 7.1

The bond pays annual coupons of 8 per \$100 of par value:

$$0.08 \times 100 = 8.00$$

The price of the bond per \$100 of par is:

$$P = \sum CF_t \left(1 + \frac{y}{m}\right)^{-mt} = 8a_{\overline{4}|5\%} + \frac{100}{1.05^4} = 110.638$$

Using this price in the formula for modified duration, we have:

$$ModD = \frac{\sum tCF_t \left(1 + \frac{y}{m}\right)^{-mt}}{P\left(1 + \frac{y}{m}\right)} = \frac{\frac{1 \times 8}{1.05} + \frac{2 \times 8}{1.05^2} + \frac{3 \times 8}{1.05^3} + \frac{4 \times 8}{1.05^4} + \frac{4 \times 100}{1.05^4}}{110.638 \times 1.05} = \frac{8(Ia)_{\overline{4}|5\%} + \frac{4 \times 100}{1.05^4}}{110.638 \times 1.05}$$

In order to calculate the present value of the increasing annuity-immediate, we first calculate the present value of an annuity-due:

$$\ddot{a}_{\overline{4}|5\%} = (1.05)\frac{1 - (1.05)^{-4}}{0.05} = 3.7232$$

$$(Ia)_{\overline{4}|5\%} = \frac{3.7232 - 4(1.05)^{-4}}{0.05} = 8.6488$$

Using the present value of the increasing annuity-immediate, we have:

$$ModD = \frac{8(Ia)_{\overline{4}|5\%} + \frac{4 \times 100}{1.05^4}}{110.638 \times 1.05} = \frac{8 \times 8.6488 + 329.0810}{116.1697} = 3.428$$

Solution 7.2

We are not given the compounding frequency of the yield, but it is not needed to answer the question.

Using the relationship between price and modified duration, we have:

$$\%\Delta P \approx -(\Delta y)(ModD)$$
$$\%\Delta P \approx -(0.001)(7.02)$$
$$\%\Delta P \approx -0.702\%$$

When the yield increases to 6.10%, the price declines by 0.702%:

$$114.72(1-0.00702) = \$113.915$$

Solution 7.3

Since there is only one cash flow, we can calculate the modified duration without calculating the price of the bond:

$$ModD = \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-mt}}{P\left(1+\frac{y}{m}\right)} = \frac{20(1{,}500)(1.02)^{-40}}{(1{,}500)(1.02)^{-40}(1.02)} = \frac{20}{1.02} = 19.608$$

Solution 7.4

The bond pays semiannual coupons of \$4 per \$100 of par value, and the semiannual yield is 5%:

$$\frac{0.08}{2}\times 100 = 4 \qquad \text{and} \qquad \frac{0.10}{2} = 0.05$$

The price of the bond per 100 of par is:

$$P = \sum CF_t\left(1+\frac{y}{m}\right)^{-mt} = 4a_{\overline{4}|5\%} + \frac{100}{1.05^4} = 4(3.5460) + 82.2702 = 96.4540$$

Using the price in the formula for modified duration, we have:

$$ModD = \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-mt}}{P\left(1+\frac{y}{m}\right)} = \frac{\frac{0.5\times 4}{1.05} + \frac{1\times 4}{1.05^2} + \frac{1.5\times 4}{1.05^3} + \frac{2\times 104}{1.05^4}}{96.4540\times 1.05}$$

$$= \frac{2(Ia)_{\overline{4}|5\%} + \frac{2\times 100}{1.05^4}}{96.4540\times 1.05} = \frac{2(8.6488) + 164.5405}{101.2768} = 1.7955$$

Solution 7.5

The Macaulay duration of a zero-coupon bond is equal to its maturity:

$$MacD = \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-tm}}{P} = \frac{15\times 2{,}000\left(1+\frac{0.07}{12}\right)^{12\times 15}}{2{,}000\left(1+\frac{0.07}{12}\right)^{12\times 15}} = 15$$

Solution 7.6

The modified duration is the Macaulay duration divided by one plus the periodic yield:

$$ModD = \frac{MacD}{\left(1+\dfrac{y}{m}\right)} = \frac{15}{1+\dfrac{0.07}{12}} = 14.913$$

Solution 7.7

The modified duration is 9.8844:

$$ModD = \frac{MacD}{\left(1+\dfrac{y}{m}\right)} = \frac{10.774}{1.09} = 9.8844$$

Using the relationship between price and modified duration, we have:

$$\%\Delta P \approx -(\Delta y)(ModD)$$
$$\%\Delta P \approx -(-0.0005)(9.8844)$$
$$\%\Delta P \approx 0.494\%$$

If the yield falls to 8.95%, then the price increases by approximately 0.494%:

$$81.12(1+0.00494) = \$81.52$$

Solution 7.8

Since the coupon rate is equal to the bond's yield, the bond is priced at par, and the Macaulay duration is therefore equal to the present value of a 30-year annuity-due:

We can either work annually or half-yearly:

Annually

$$MacD = \ddot{a}_{\overline{n|}}^{(m)} = \ddot{a}_{\overline{30|}}^{(2)} = \left(1+\frac{i^{(2)}}{2}\right)a_{\overline{30|}}^{(2)} = 1.03\left(\frac{1-\left(1.03^2\right)^{-30}}{0.06}\right) = 14.253$$

Half-yearly

$$\ddot{a}_{\overline{30|}}^{(2)} = \frac{1}{2}\left(1+\frac{i^{(2)}}{2}\right)a_{\overline{60|}} = \frac{1}{2}(1.03)\left(\frac{1-(1.03)^{-60}}{0.03}\right) = 14.253$$

The modified duration is thus:

$$ModD = \frac{MacD}{\left(1+\dfrac{y}{m}\right)} = \frac{14.253}{1.03} = 13.838$$

Solution 7.9

The monthly effective yield is $\frac{0.12}{12} = 0.01$. The amount of the monthly payment is denoted below as Pmt. Notice that we do not need to determine the amount of the monthly payment to calculate the duration. The first monthly cash flow occurs at time $t = \frac{1}{12}$, the second monthly cash flow occurs at time $t = \frac{2}{12}$, and so on:

$$MacD = \frac{\sum tCF_t \left(1 + \frac{y}{m}\right)^{-tm}}{P} = \frac{\frac{1 \times Pmt}{12(1.01)^1} + \frac{2 \times Pmt}{12(1.01)^2} + \cdots + \frac{180 \times Pmt}{12(1.01)^{180}}}{\frac{Pmt}{(1.01)^1} + \frac{Pmt}{(1.01)^2} + \cdots + \frac{Pmt}{(1.01)^{180}}}$$

$$= \frac{\frac{Pmt}{12} \times (Ia)_{\overline{180}|1\%}}{Pmt \times a_{\overline{180}|1\%}} = \frac{\frac{1}{12}(Ia)_{\overline{180}|1\%}}{a_{\overline{180}|1\%}} = \frac{\frac{1}{12} \times 5,413.3876}{83.3217} = 5.414$$

Solution 7.10

The effective duration is 8.81:

$$EffD = \frac{P_- - P_+}{P_0(2\Delta y)} = \frac{130.65 - 125.02}{127.79(2)(0.0025)} = 8.81$$

Solution 7.11

The effective duration is:

$$EffD = \frac{P_- - P_+}{P_0(2\Delta y)} = \frac{101.6931 - 100.8422}{(101.2666)(2)(0.0010)} = 4.20$$

Solution 7.12

The effective convexity is 20.74:

$$EffC = \frac{(P_+ + P_- - 2P_0)}{(\Delta y)^2 P_0} = \frac{100.8422 + 101.6931 - 2(101.2666)}{(0.001)^2(101.2666)} = 20.74$$

Solution 7.13

Since the yield increases, the percentage change in price is negative:

$$\begin{aligned}
\%\Delta P &\approx -(\Delta y)(\text{Duration}) + \frac{(\Delta y)^2}{2}(\text{Convexity}) \\
&= -(0.0075)(4.20) + \frac{(0.0075)^2}{2}(20.74) \\
&= -0.0315 + 0.0005833 \\
&= -3.09167\%
\end{aligned}$$

The new price is \$98.14:

$$\text{New price} = (1 - 0.0309167)101.2666 = 98.14$$

Solution 7.14

The general expression for the price of a perpetuity-immediate paying \$1 at the end of each year is:

$$P(y)=\frac{1}{y}$$

where:

y = annual effective yield

It is a simple matter to find the first and second derivatives of the price of the perpetuity-immediate:

$$P'(y)=-\frac{1}{y^2}$$

$$P''(y)=2\frac{1}{y^3}$$

Now we can derive general expressions for the modified duration and convexity of the perpetuity:

$$ModD=-\frac{P'(y)}{P(y)}=\frac{1}{y}$$

$$\text{Convexity}=\frac{P''(y)}{P(y)}=\frac{2}{y^2}$$

Interestingly, the modified duration of a perpetuity-immediate paying \$1 at the end of each year is equal to its price:

$$\text{Price}=\frac{1}{0.05}=20.0$$

$$ModD=\frac{1}{0.05}=20.0$$

$$\text{Convexity}=\frac{2}{0.05^2}=800.0$$

Solution 7.15

When there is just one cash flow, the current yield of the bond is not needed in order to calculate Macaulay duration and Macaulay convexity:

$$MacD=\frac{\sum tCF_t e^{-\delta t}}{P}=\frac{5CF_5 e^{-5\delta}}{CF_5 e^{-5\delta}}=5$$

$$MacC=\frac{\sum t^2 CF_t e^{-\delta t}}{P}=\frac{25CF_5 e^{-5\delta}}{CF_5 e^{-5\delta}}=25$$

For zero-coupon bonds, the Macaulay convexity is the square of the Macaulay duration.

Solution 7.16

Since the yield increases, the percentage change in price is negative:

$$\begin{aligned}
\%\Delta P &\approx -(\Delta y)(\text{Duration}) + \frac{(\Delta y)^2}{2}(\text{Convexity}) \\
&= -(0.0063)5.35 + \frac{(0.0063)^2}{2}(39.19) \\
&= -0.033705 + 0.000778 \\
&= -3.29\%
\end{aligned}$$

Solution 7.17

The present value of the assets and the present value of the liabilities are:

$$PV_A = 27{,}919.74 + 27{,}919.74 = 55{,}839.48$$

$$PV_L = \frac{100{,}000}{1.06^{10}} = 55{,}839.48$$

The first immunization condition, that the present value of the assets must equal the present value of the liabilities, is satisfied.

The Macaulay duration of the assets and the Macaulay duration of the liabilities are:

$$MacD_A = 0.5(5) + 0.5(15) = 10$$

$$MacD_L = 10$$

The second immunization condition, that the duration of the assets be equal to the duration of the liabilities, is satisfied.

In Solution 7.15, we saw that the Macaulay convexity of a zero-coupon bond is equal to the square of its Macaulay duration. Therefore, the Macaulay convexities are:

$$\text{Convexity of 5-year bond} = 5^2 = 25$$

$$\text{Convexity of 15-year bond} = 15^2 = 225$$

$$MacC_A = 0.5(25) + 0.5(225) = 125$$

$$MacC_L = 10^2 = 100$$

The third immunization condition, that the convexity of the assets must be greater than the convexity of the liabilities is satisfied.

The company's position is therefore immunized.

Solution 7.18

The total amount invested must equal the present value of the liability:

$$PV_L = \frac{100,000}{1.12^5} = 56,742.69$$

The Macaulay duration of the liability is 5:

$$MacD_L = 5$$

If $x\%$ of the \$56,742.69 is invested in the 4-year bond and $(1-x\%)$ is invested in the 10-year bond, we use immunization condition #2 to solve for $x\%$:

$$\begin{aligned}&MacD_A = MacD_L\\&(x\%)4+(1-x\%)10 = 5\\&-6x\% = -5\\&x\% = 83.33\% \quad \text{and} \quad 1-x\% = 16.67\%\end{aligned}$$

Checking to see if immunization condition #3 is satisfied:

$$\begin{aligned}&MacC_L = 5^2 = 25\\&MacC_A = 0.8333(4)^2 + 0.1667(10)^2 = 30\end{aligned}$$

Since the convexity of the assets is greater than the convexity of the liabilities, the company's position is immunized.

To establish its asset portfolio, the company invests \$47,285.58 in the 4-year bond and \$9,457.12 in the 10-year bond:

$$\begin{aligned}&0.8333 \times 56,742.69 = 47,285.58\\&0.1667 \times 56,742.69 = 9,457.12\end{aligned}$$

Solution 7.19

Modified duration and effective duration are calculated below:

$$\begin{aligned}ModD &= \frac{\sum tCF_t\left(1+\frac{y}{m}\right)^{-mt}}{P\left(1+\frac{y}{m}\right)}\\&= \frac{\frac{8}{1.07}+\frac{2(7.9)}{(1.07)^2}+\frac{3(107.80)}{(1.07)^3}}{102.37(1.07)} = \frac{285.2677}{102.37(1.07)} = 2.6043\end{aligned}$$

$$EffD = \frac{P_- - P_+}{P_0(2\Delta y)} = \frac{104.33-99.76}{(102.37)(2)(0.01)} = 2.2321$$

The ratio of modified duration to effective duration is 1.167:

$$\frac{2.6043}{2.2321} = 1.167$$

Solution 7.20

Modified duration is the derivative of price with respect to the yield, divided by price:

$$ModD = -\frac{P'(y)}{P(y)} = -\frac{-700}{100} = 7$$

We are not told the compounding frequency of the 8% yield. When we are not given the compounding frequency of a yield, it is customary to assume that it has the same compounding frequency as the coupons. When, as in this case, we are also not given the compounding frequency of the coupons, it is customary to assume that the yield is an annual effective yield:

$$ModD = \frac{MacD}{\left(1+\frac{y}{m}\right)} \quad \Rightarrow \quad 7 = \frac{MacD}{1.08} \quad \Rightarrow \quad MacD = 7(1.08) = 7.56$$

The Macaulay duration is 7.56.

