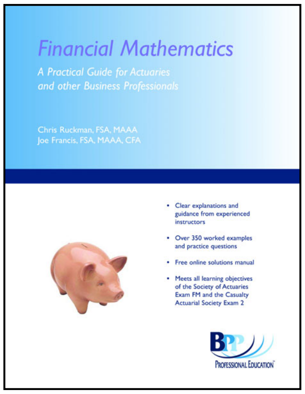

BPP PROFESSIONAL EDUCATION

Financial Mathematics

A Practical Guide for Actuaries and other Business Professionals

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Solutions to practice questions – Chapter 9

Solution 9.1

The table below describes the probability distribution of the accumulated amount at the end of 10 years:

$1,000AV_{10}$	Probability
$1,000\times1.05\times1.05^9=1,628.8946$	0.25
$1,000\times1.05\times1.07^9=1,930.3822$	0.40
$1,000\times1.05\times1.10^9=2,475.8451$	0.35

The expected accumulated value is:

$$1,628.8946\times0.25+1,930.3822\times0.40+2,475.8451\times0.35=\$2,045.92$$

Solution 9.2

The table below shows the probability for each possible value of $(1,000AV_{10})^2$.

$1,000AV_{10}$	$(1,000AV_{10})^2$	Probability
1,628.8946	$1,628.8946^2=2,653,297.71$	0.25
1,930.3822	$1,930.3822^2=3,726,375.33$	0.40
2,475.8451	$2,475.8451^2=6,129,808.84$	0.35

The expected value of the square of the accumulated value after 10 years is:

$$E[(1,000AV_{10})^2]=0.25\times2,653,297.71+0.40\times3,726,375.33+0.35\times6,129,808.84=4,299,307.65$$

The standard deviation of the accumulated value is:

$$\text{Standard deviation} = \left[E[(1{,}000AV_{10})^2] - (E[1{,}000AV_{10}])^2\right]^{0.5} = \left[4{,}299{,}307.65 - 2{,}045.92^2\right]^{0.5} = \$336.91$$

Solution 9.3

The mean and the variance of the annual yield are:

$$\bar{i} = E[i_t] = (0.2)(0.10) + (0.5)(0.12) + (0.3)(0.15) = 0.125$$

$$s^2 = (0.2)(0.10^2) + (0.5)(0.12^2) + (0.3)(0.15^2) - 0.125^2 = 0.000325$$

The mean of the accumulated amount at the end of 6 years is:

$$E[5{,}000AV_6] = 5{,}000 \times E[AV_6] = 5{,}000(1.125)^6 = \$10{,}136.43$$

The variance is:

$$\begin{aligned} Var(5{,}000AV_6) &= 5{,}000^2\,Var(AV_6) \\ &= 5{,}000^2\left[[(1+\bar{i})^2 + s^2]^6 - (1+\bar{i})^{2\times 6}\right] \\ &= 5{,}000^2\left[[(1.125)^2 + 0.000325]^6 - (1.125)^{12}\right] \\ &= 158{,}408.56 \end{aligned}$$

The standard deviation is therefore:

$$\text{Standard deviation} = \sqrt{158{,}408.56} = \$398.01$$

Solution 9.4

The mean of the accumulated value at the end of three years is:

$$E[100{,}000AV_3] = 100{,}000E[AV_3] = 100{,}000(1.14)^3 = \$148{,}154.40$$

Solution 9.5

Statements III and IV are the only correct statements.

I. $E[AV_n] = E\left[1 + i^n\right]$

II. $E[AV_n] = E\left[(1+i)^n\right]$

III. $E[AV_n] = \left(E[1+i]\right)^n$

IV. $E[AV_n] = \left(1 + E[i]\right)^n$

Statement I is clearly not correct since its value goes to one as n increases (assuming that the possible values of i are positive values less than 1).

Statement II is the formula for the fixed interest rate model.

Statement III simplifies to the correct formula for independent and identically distributed interest rates:

$$E[AV_n] = \left(E[1+i]\right)^n = \left(1 + E[i]\right)^n = \left(1 + \bar{i}\right)^n$$

Statement IV also simplifies to the correct formula for independent and identically distributed interest rates:

$$E[AV_n] = (1+E[i])^n = \left(1+\bar{i}\right)^n$$

Solution 9.6

The expected value of the accumulated value is:

$$\begin{aligned}\text{Expected value} &= E\left[10{,}000(1+i_1)(1+i_2)(1+i_3) + 8{,}000(1+i_2)(1+i_3) + 32{,}000(1+i_3)\right] \\ &= 10{,}000E(1+i_1)E(1+i_2)E(1+i_3) + 8{,}000E(1+i_2)E(1+i_3) + 32{,}000E(1+i_3) \\ &= 10{,}000(1.04)(1.10)(1.07) + 8{,}000(1.10)(1.07) + 32{,}000(1.07) \\ &= \$55{,}896.80\end{aligned}$$

Solution 9.7

Since the interest rates are independent and identically distributed, we have:

$$\begin{aligned}E[s_{\overline{n|}}] &= E\left[(1+i_2)(1+i_3)\cdots(1+i_n)\right] + E\left[(1+i_3)(1+i_4)\cdots(1+i_n)\right] + \cdots + E\left[(1+i_n)\right] + E[1] \\ &= \left[E(1+i_2)E(1+i_3)\cdots E(1+i_n)\right] + \left[E(1+i_3)E(1+i_4)\cdots E(1+i_n)\right] + \cdots + \left[E(1+i_n)\right] + E[1] \\ &= (1+\bar{i})^{n-1} + (1+\bar{i})^{n-2} + \cdots(1+\bar{i}) + 1 \\ &= s_{\overline{n|}\bar{i}}\end{aligned}$$

If the interest rates are independent and identically distributed, then the expected accumulated value of an annuity immediate is equal to an annuity immediate calculated at the mean interest rate.

Solution 9.8

Part (i)

$$E[AV_2] = (1+\bar{i})^2 = (1.05)^2 = \$1.1025$$

Part (ii)

$$\begin{aligned}Var(AV_2) &= [(1+\bar{i})^2 + s^2]^2 - (1+\bar{i})^{2\times 2} \\ &= [(1.05)^2 + 0.0002]^2 - (1.05)^4 \\ &= 0.000441\end{aligned}$$

Solution 9.9

We can use the formula for the expected value of the 1-year accumulation factor to find the value of $\mu+\sigma^2/2$:

$$E[1+i_t]=e^{\mu+\sigma^2/2}$$
$$1.05=e^{\mu+\sigma^2/2}$$
$$\mu+\sigma^2/2=\ln(1.05)$$

This can then be put into the formula for the variance to solve for σ^2:

$$Var[1+i_t]=e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$
$$0.0002=e^{2(\mu+\sigma^2/2)}(e^{\sigma^2}-1)$$
$$0.0002=e^{2\ln(1.05)}(e^{\sigma^2}-1)$$
$$0.0002=1.05^2(e^{\sigma^2}-1)$$
$$e^{\sigma^2}=0.0002/1.05^2+1$$
$$\sigma^2=0.0001814$$

Now we can find μ:

$$\mu+\sigma^2/2=\ln(1.05)$$
$$\mu=\ln(1.05)-0.0001814/2=0.0486995$$

Solution 9.10

Part (i)

The general formula is:

$$E[AV_n]=e^{n\mu+n\sigma^2/2}$$

For $n=2$, we have:

$$\begin{aligned}E[AV_2]&=e^{2\mu+\sigma^2}\\&=e^{2(0.0486995)+(0.0001814)}\\&=1.1025\end{aligned}$$

Part (ii)

The general formula is:

$$Var[AV_n]=e^{2n\mu+n\sigma^2}(e^{n\sigma^2}-1)$$

For $n=2$, we have:

$$\begin{aligned}Var[AV_2]&=e^{4\mu+2\sigma^2}(e^{2\sigma^2}-1)\\&=e^{4(0.0486995)+2(0.0001814)}(e^{2(0.0001814)}-1)\\&=0.000441\end{aligned}$$

As expected, the expected accumulated value and the variance are the same as the expected accumulated value and variance found in Solution 9.8, since both stochastic models are independent and identically distributed. Even if their underlying distributions are different from one another, the fact that the interest rates have the same mean and variance means that the mean and variance of the accumulated values are also the same.

Solution 9.11

Only statements I and III are correct.

We are given that:

$$E[(1+i_t)] = 1.10 \quad \text{and} \quad Var[1+i_t] = Var[i_t] = 0.025^2$$

Furthermore, we are given that $(1+i_t)$ has a lognormal distribution.

Statement I is correct because:

$$E[AV_2] = (1.10)^2 = 1.21$$

Statement II is not correct because:

$$\begin{aligned} Var(AV_2) &= [(1+\bar{i})^2 + s^2]^2 - (1+\bar{i})^{2\times 2} \\ &= \left[1.10^2 + 0.025^2\right]^2 - 1.10^4 \\ &= 0.00151289 \end{aligned}$$

The standard deviation is therefore:

$$\sqrt{0.00151289} = 0.0388959$$

Statement III is correct because when the accumulation factors in different years have independent and identical lognormal distributions, the accumulated value of multiple years also has a lognormal distribution.

Solution 9.12

Part (i)

We use μ to denote the mean and σ^2 to denote the variance of $\ln(1+i_t)$.

$$\mu = E[\ln(1+i_t)] = 0.5\ln(1.08) + 0.5\ln(1.12) = 0.095145$$

$$E[\ln(1+i_t)^2] = 0.5[\ln(1.08)]^2 + 0.5[\ln(1.12)]^2 = 0.0093832$$

$$\sigma^2 = Var[\ln(1+i_t)] = 0.0093832 - (0.095145)^2 = 0.00033065$$

Part (ii)

The distribution of $\ln(AV_n)$ approaches a normal distribution as n increases:

$$\begin{aligned} P(10{,}000AV_{40} > 400{,}000) = P(AV_{40} > 40) &= P\left[Z > \frac{\ln 40 - n\mu}{\sigma\sqrt{n}}\right] = P\left[Z > \frac{\ln 40 - 40(0.095145)}{\sqrt{0.00033065}\sqrt{40}}\right] \\ &= P\left[Z > -1.016612\right] = P\left[Z < 1.016612\right] = 0.845 \end{aligned}$$

The probability of 84.5% was obtained from a standard normal table. The probability of the fund being at least \$400,000 is 84.5%.

Solution 9.13

We are given that:

$$E[\ln(1+i_t)] = \mu = 0.076911$$
$$Var[\ln(1+i_t)] = \sigma^2 = 0.0001$$

The lower limit of the confidence interval is obtained by recalling that the 95% confidence interval for a standard normal random variable is $(-1.96, 1.96)$. Below we find the 95% confidence interval for $100AV_3$, with X being the lower limit and Y being the upper limit:

$$P(100AV_3 < X) = \frac{0.05}{2}$$
$$P(AV_3 < X/100) = 0.025$$
$$P\left[Z < \frac{\ln(X/100) - 3\mu}{\sigma\sqrt{3}}\right] = 0.025$$
$$P\left[Z < \frac{\ln(X/100) - 3(0.076911)}{\sqrt{0.0001}\sqrt{3}}\right] = 0.025$$
$$\frac{\ln(X/100) - 3(0.076911)}{\sqrt{0.0001}\sqrt{3}} = -1.96$$
$$X = \$121.748$$

Likewise:

$$P(100AV_3 > Y) = 0.025$$
$$P(AV_3 > Y/100) = 0.025$$
$$P\left[Z > \frac{\ln(Y/100) - 3\mu}{\sigma\sqrt{3}}\right] = 0.025$$
$$P\left[Z > \frac{\ln(Y/100) - 3(0.076911)}{\sqrt{0.0001}\sqrt{3}}\right] = 0.025$$
$$\frac{\ln(Y/100) - 3(0.076911)}{\sqrt{0.0001}\sqrt{3}} = 1.96$$
$$Y = \$130.302$$

The 95% confidence interval for $\$100AV_3$ is ($121.75, $130.30).

Solution 9.14

The mean and standard deviation are:

$$E[i_t] = E[1+i_t] - 1 = e^{\mu+\sigma^2/2} - 1 = e^{0.06+0.001/2} - 1 = 1.0623676 - 1 = 0.0623676$$
$$Var[i_t] = Var[1+i_t] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) = e^{2(0.06)+0.001}(e^{0.001} - 1) = 0.0011292$$

Solution 9.15

The probability of the interest rates being less than the median is 50%. Let's denote the median as M:

$$P(i_t < M) = 0.50$$
$$P(1+i_t < 1+M) = 0.50$$
$$P[\ln(1+i_t) < \ln(1+M)] = 0.50$$
$$P\left[\frac{\ln(1+i_t)-0.06}{\sqrt{0.001}\sqrt{1}} < \frac{\ln(1+M)-0.06}{\sqrt{0.001}\sqrt{1}}\right] = 0.50$$
$$P\left[Z < \frac{\ln(1+M)-0.06}{\sqrt{0.001}}\right] = 0.50$$

Since $P[Z<0] = 0.50$, we have:

$$\frac{\ln(1+M)-0.06}{\sqrt{0.001}} = 0$$
$$M = 0.061837$$

The median is 6.1837%.

Solution 9.16

We want to know the probability that the accumulated value of the fund is greater than or equal to $10,000 in five years:

$$P[7{,}000 AV_5 > 10{,}000]$$
$$= P[AV_5 > 1.42857]$$
$$= P[\ln(AV_5) > \ln(1.42857)]$$
$$= P\left[\frac{\ln(AV_5)-5(0.06)}{\sqrt{0.001\times 5}} > \frac{\ln(1.42857)-5(0.06)}{\sqrt{0.001\times 5}}\right]$$
$$= P\left[Z > 0.8015\right]$$

From a standard normal table, we find that $P\left[Z > 0.8015\right] = 0.211$, so the probably that the accumulated value will be sufficient to meet the liability is 21.1%.

Solution 9.17

There are two methods for answering the question.

Method 1

We can determine the interest rate needed under the constant model to accumulate \$7,000 to \$10,000 over 5 years:

$$7{,}000(1+i)^5 = 10{,}000$$
$$i = 0.07394$$

Under the constant model, the probability of the interest rate being greater than 7.394% is equal to the probability of the \$7,000 accumulating to \$10,000:

$$\begin{aligned}
&P[i > 0.07394] \\
&= P[1+i > 1.07394] \\
&= P[\ln(1+i) > \ln(1.07394)] \\
&= P\left[\frac{\ln(1+i)-0.06}{\sqrt{0.001}} > \frac{\ln(1.07394)-0.06}{\sqrt{0.001}}\right] \\
&= P\left[Z > 0.35844\right]
\end{aligned}$$

Using a standard normal table:

$$P\left[Z > 0.35844\right] = 36.0\%$$

Method 2

Alternatively, we can find the mean and variance of AV_5 under the constant model.

$$AV_5 = (1+i)^5$$
$$\ln(AV_5) = 5\ln(1+i)$$

We know that $\ln(1+i)$ is a normal random variable described by $\sim N(0.06, 0.001)$:

$$\ln(1+i) \sim N(0.06, 0.001)$$
$$5\ln(1+i) \sim N(5\times 0.06, 5^2 \times 0.001)$$
$$\ln(AV_5) \sim N(0.30, 0.025)$$

Thus, under the constant model, AV_5 is lognormal with parameters 0.30 and 0.025. The probability that \$7,000 will grow to \$10,000 in five years is:

$$\begin{aligned}
&P[7{,}000 AV_5 > 10{,}000] \\
&= P[AV_5 > 1.42857] \\
&= P[\ln(AV_5) > \ln(1.42857)] \\
&= P\left[\frac{\ln(AV_5)-5(0.06)}{\sqrt{0.025}} > \frac{\ln(1.42857)-5(0.06)}{\sqrt{0.025}}\right] \\
&= P\left[Z > 0.35844\right]
\end{aligned}$$

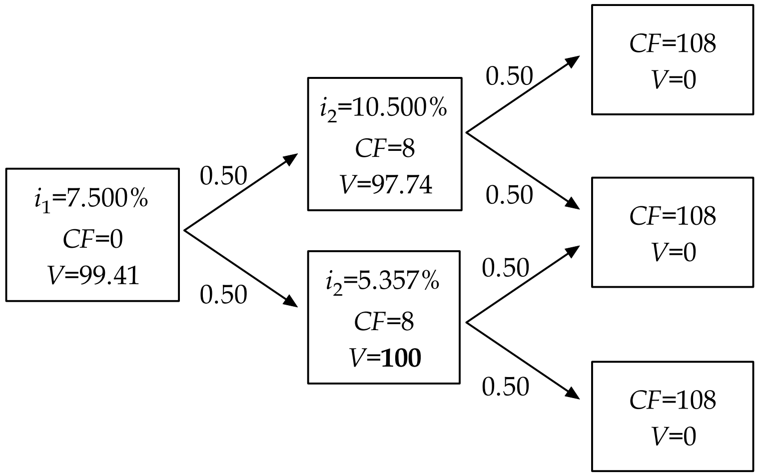
Using a standard normal table:

$$P\left[Z > 0.35844\right] = 36.0\%$$

Therefore, under the constant model the probability of the \$7,000 accumulating to more than \$10,000 in five years is 36.0%. This is higher than the 21.1% found when we assumed that the interest rates in each year are independent.

Solution 9.18

The cash flows and values at the nodes are shown below.



Beginning with the nodes at time 2 we can work backwards, valuing one node at a time with the following formula:

$$V = \frac{p(V_U + CF_U) + (1-p)(V_D + CF_D)}{(1+i_t)}$$

Applying the formula:

Upper node at time 1: $V = \dfrac{0.5(0+108)+0.5(0+108)}{1.105} = \$97.74$

Lower node at time 1: $V = \dfrac{0.5(0+108)+0.5(0+108)}{1.05357} = \$102.51$

but this value cannot exceed the call price of \$100, so:

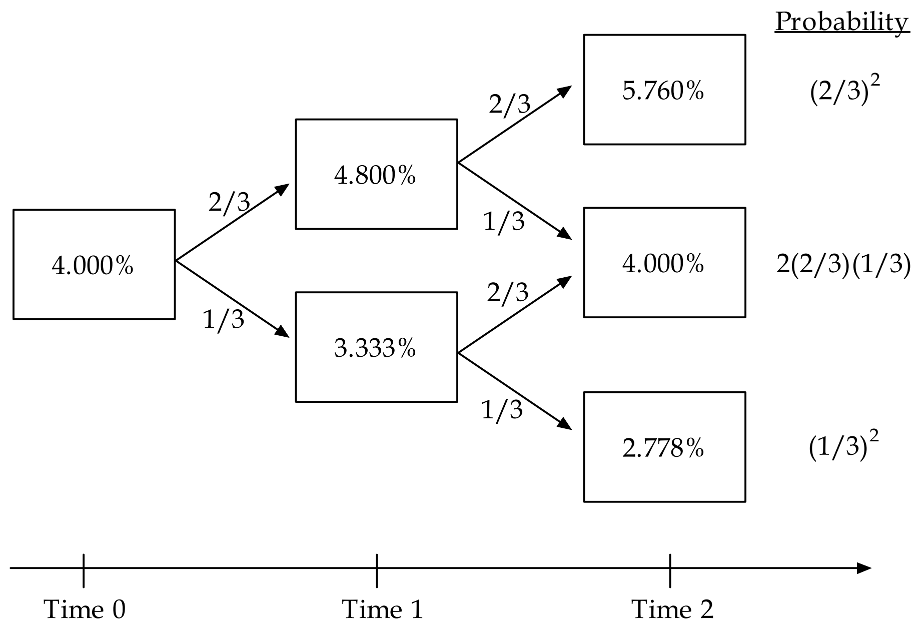
$V = \$100$

Value at time 0: $V = \dfrac{0.5(8+97.74)+0.5(8+100)}{1.075} = \$99.41$

The value of the callable bond is \$99.41.

Solution 9.19

The interest rates are twice as likely to rise as they are to fall. Since the probabilities of rising and falling must sum to 1, we have $p = 2/3$.



The short rate two periods from now is the rate used to discount a cash flow from time 3 to time 2. Multiplying the probability of each possible short rate in two periods by its probability gives us the expected value:

$$\left(\frac{2}{3}\right)^2(0.05760) + 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)(0.04000) + \left(\frac{1}{3}\right)^2(0.02778) = 0.0465$$

The expected value of the short rate two periods from now is 4.65%.

Solution 9.20

Notice that the notation in this question varies from the convention followed in the text of the chapter. For this question, r_t denotes an interest rate that applies from time t to time $(t+1)$.

Recall that an interest rate floor pays:

$$\text{Floor payment} = \text{Max}\left[\text{Strike rate} - \text{Index rate, } 0\right] \times \text{Notional amount}$$

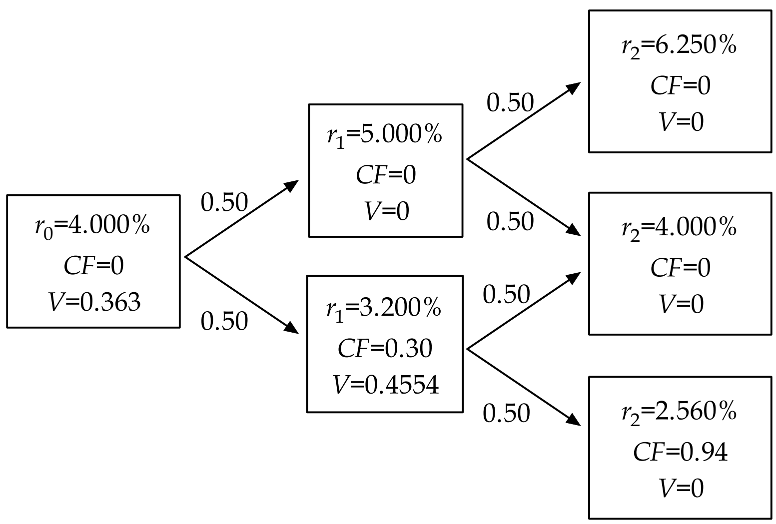
At the lowest node at time 3, the cash flow is:

$$\text{Floor payment} = (0.035 - 0.0256)(100) = \$0.94$$

The only other positive cash flow occurs at time 2 in the lowest node:

$$\text{Floor payment} = (0.035 - 0.0320)(100) = \$0.30$$

The cash flows and values at the nodes are shown below.



Beginning with the nodes at time 2 we can work backwards, valuing one node at a time with the following formula:

$$V = \frac{p(V_U + CF_U) + (1-p)(V_D + CF_D)}{(1 + r_{t-1})}$$

Applying the formula:

Upper node at time 1: $$V = \frac{0.5(0+0) + 0.5(0+0)}{1.05} = \$0$$

Lower node at time 1: $$V = \frac{0.5(0+0) + 0.5(0+0.94)}{1.032} = \$0.4554$$

Value at time 0: $$V = \frac{0.5(0+0) + 0.5(0.30+0.4554)}{1.04} = \$0.363$$

The value of the interest rate floor is \$0.363.