Construction of Actuarial Models
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Solutions to practice questions – Chapter 6

Solution 6.1

With 6 degrees of freedom, the 95th percentile is 12.592 and the 97.5th percentile is 14.449. Since the statistic value 13.5 is between these two percentiles we can say that the p-value is between 2.5% and 5%.

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Solution 6.2

The null and alternative hypotheses are:

H_0 : the Binomial $(m = 4, q = 0.5)$ model is correct

H_1 : the Binomial $(m = 4, q = 0.5)$ model is not correct

Under the suggested model the expected frequencies are:

Number of boys	0	1	2	3	4	Total
Expected frequency	9.375	37.5	56.25	37.5	9.375	150

These figures have been calculated using the formula:

$$150 \Pr(N = n) = 150 \binom{4}{n} 0.5^n \, 0.5^{4-n} = 150 \binom{4}{n} 0.5^4$$

The test statistic is:

$$\chi^2 = \frac{(8-9.375)^2}{9.375} + \frac{(36-37.5)^2}{37.5} + \frac{(61-56.25)^2}{56.25} + \frac{(32-37.5)^2}{37.5} + \frac{(13-9.375)^2}{9.375}$$

$$= 2.871$$

We compare this with χ_4^2. The upper 5% point of χ_4^2 is 9.49. As the test statistic is less than this critical value, we have insufficient evidence to reject the null hypothesis and we conclude that the binomial model is correct.

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Solution 6.3

We would have compared the value of the test statistic with χ_3^2, instead of χ_4^2.

Solution 6.4

The following values are realizations of a random variable Y:

      100   150   225   290   300   500

You want to test whether these data come from a Pareto distribution with parameters $\alpha = 3$ and $\theta = 600$.

The hypothesized CDF is:

$$F(y) = 1 - \left(\frac{600}{y+600}\right)^3$$

For the given data we have:

j	$y_{(j)}$	$F\left(y_{(j)}\right)$	$\frac{j}{6} - F\left(y_{(j)}\right)$	$F\left(y_{(j)}\right) - \frac{j-1}{6}$
1	100	0.370	−0.204	0.370
2	150	0.488	−0.155	0.321
3	225	0.615	−0.115	0.282
4	290	0.694	−0.027	0.194
5	300	0.703	0.130	0.037
6	500	0.838	0.162	0.004

So the Kolmogorov-Smirnov test statistic is 0.370.

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Solution 6.5

The geometric distribution is a special case of the negative binomial distribution with the restriction that $r = 1$. As we are given the maximized log-likelihood values, the likelihood ratio test statistic is:

$$-2(-578.9 + 573.6) = 10.6$$

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Solution 6.6

Using the suggested percentiles, we have the following results:

Range	Observed	Expected
0 - 1,000	3	4
1,000 - 3,000	7	8
3,000 - 5,000	10	6
5,000 - 8,000	9	10
8,000 - 11, 000	5	8
11,000 +	6	4

Combining the first two groups and the last two groups to make the expected values at least 5, we obtain the following test statistic:

$$\chi^2 = \frac{(10-12)^2}{12} + \frac{(10-6)^2}{6} + \frac{(9-10)^2}{10} + \frac{(11-12)^2}{12} = 3.183$$

Solution 6.7

We first have to calculate the cdf. If $0 < x \le \frac{1}{2}$, then:

$$F(x) = \int_0^x 4t \, dt = \left[2t^2 \right]_0^x = 2x^2$$

If $\frac{1}{2} < x < 1$, then:

$$F(x) = \frac{1}{2} + \int_{\frac{1}{2}}^x (4 - 4t) \, dt = \frac{1}{2} + \left[4t - 2t^2 \right]_{\frac{1}{2}}^x = \frac{1}{2} + \left(4x - 2x^2 \right) - \left(2 - \frac{1}{2} \right) = 4x - 2x^2 - 1$$

So we have:

j	$x_{(j)}$	$F(x_{(j)})$	$\frac{j}{4} - F(x_{(j)})$	$F(x_{(j)}) - \frac{j-1}{4}$
1	0.16	0.0512	0.1988	0.0512
2	0.39	0.3042	0.1958	0.0542
3	0.62	0.7112	0.0388	0.2112
4	0.75	0.8750	0.1250	0.1250

The maximum of the numbers in the last two columns is 0.2112, and this is the Kolmogorov-Smirnov test statistic.

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Solution 6.8

The ranking criterion is $\ln L - \dfrac{r}{2} \ln n$, where r is the number of parameters in the model and n is the number of data points. The figures are as follows:

Inverse Burr model: $\ln(L) - \dfrac{r}{2} \ln(n) = -180.2 - \dfrac{3}{2} \ln(200) = -188.1$

Inverse Gamma model: $\ln(L) - \dfrac{r}{2} \ln(n) = -181.4 - \dfrac{2}{2} \ln(200) = -186.7$

Weibull model: $\ln(L) - \dfrac{r}{2} \ln(n) = -181.6 - \dfrac{2}{2} \ln(200) = -186.9$

Exponential model: $\ln(L) - \dfrac{r}{2} \ln(n) = -183.0 - \dfrac{1}{2} \ln(200) = -185.6$

The largest of these numbers is –185.6, so the best model according to the Schwarz-Bayesian criterion is the Exponential.

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Solution 6.9

The observation 267 is the sixth out of seven when they are put in increasing order. The 2-parameter Pareto CDF is:

$$F(x) = 1 - \left(\frac{\theta}{\theta + x} \right)^{\alpha} = 1 - \left(\frac{250}{250 + x} \right)^2$$

So the point sought is:

$$\left(\frac{6}{7+1} , F(267) \right) = (0.750 , 0.766)$$

Solution 6.10

(i) $f(x) = \dfrac{\beta^x}{(1+\beta)^{x+1}}$ for $x=0,1,\dots$ $\Rightarrow$ $L(\beta) = \dfrac{\beta^{\sum x_i}}{(1+\beta)^{\sum(x_i+1)}}$

$\Rightarrow$ $\ln(L(\beta)) = \sum x_i \times \ln(\beta) - \sum(x_i+1)\times \ln(1+\beta)$

$\Rightarrow$ $0 = \dfrac{d\ln(L(\beta))}{d\beta} = \dfrac{\sum x_i}{\beta} - \dfrac{\sum(1+x_i)}{1+\beta} = \dfrac{9}{\beta} - \dfrac{9+8}{1+\beta}$ $\Rightarrow$ $\hat{\beta} = \dfrac{9}{8}$

(ii) The maximum value is:
$$\ln\left(L\left(\hat{\beta}\right)\right) = \sum x_i \times \ln(9/8) - \sum(x_i+1)\times \ln(1+9/8)$$
$$= 9\times \ln(9/8) - 17\times \ln(17/8) = -11.754$$

(iii) There is 1 restriction. The 95th percentile of the chi-squared distribution with 1 degree of freedom is 3.841.

So the null hypothesis is rejected if the statistic exceeds 3.841:
$$3.841 < 2\left(\ln(L_{UR}) - \ln(L_R)\right) = 2\left(\ln(L_{UR}) + 11.754\right) \iff -9.834 < \ln(L_{UR})$$