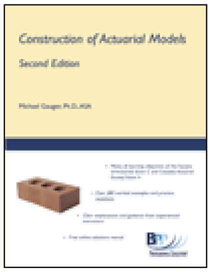

Construction of Actuarial Models

Second Edition

by Mike Gauger

Published by BPP Professional Education

Solutions to practice questions – Chapter 15

Solution 15.1

$$
\begin{aligned}
C - P &= e^{-rt}\left(E\left[(S_t-K)_+\right] - E\left[(K-S_t)_+\right]\right) \\
&= e^{-rt} \times E\left[(S_t-K)_+ - (K-S_t)_+\right] \\
&= e^{-rt} \times E\left[S_t - K\right] \quad \text{(see the Solution to Question 14.17)} \\
&= e^{-rt} \times \left(S_0 e^{(r-\delta)t} - K\right)
\end{aligned}
$$

Solution 15.2

$$
\frac{\partial\left(e^{-rt} \times \left(S_0 e^{(\alpha-\delta)t} - K\right)\right)}{\partial K} = -e^{-rt}
$$

Interpretation: a change of ΔK in the strike price causes the difference between the call and put prices to decrease by $e^{-rt}\,\Delta K$.

Solution 15.3

$$
E\left[S_{0.5}\right] = S_0\, e^{(r-\delta)0.50} = 100\ e^{0.08\times 0.50} = 104.0811
$$

$$
\begin{aligned}
l &= \frac{\ln(K/S_0) - (r-\delta+0.5\sigma^2)t}{\sigma\sqrt{t}} \\
&= \frac{\ln(100/100) - (0.08 + 0.5\times 0.20^2)\times 0.50}{0.20 \times \sqrt{0.50}} = -0.3536
\end{aligned}
$$

$$
\Rightarrow \Phi(-l) = \Phi(0.3536) = 0.6382
$$

$$u = \frac{\ln\left(\left(K/S_0\right)\right) - (r-\delta-0.5\sigma^2)t}{\sigma\sqrt{t}}$$
$$= \frac{\ln(100/100) - (0.08 - 0.5\times 0.20^2)\times 0.50}{0.20\times\sqrt{0.50}} = -0.2121$$

$$\Rightarrow \Phi(-u) = \Phi(0.2121) = 0.5840$$

$$C = e^{-rt}\left(E[S_t]\,\Phi(-l) - K\,\Phi(-u)\right)$$
$$= e^{-0.08\times 0.5}\left(104.0811\times 0.6382 - 100\times 0.5840\right) = 7.71$$

Solution 15.4

We can use some of the same calculations as in Solution 15.3:

$$E[S_{0.5}] = 104.0811$$
$$l = -0.3536 \Rightarrow \Phi(l) = 1-\Phi(0.3536) = 0.3618$$
$$u = -0.2121 \Rightarrow \Phi(u) = 1-\Phi(0.2121) = 0.4160$$
$$P = e^{-rt}\left(K\,\Phi(u) - E[S_t]\,\Phi(l)\right)$$
$$= e^{-0.08\times 0.50}\left(100\times 0.4160 - 108.0811\times 0.3618\right)$$
$$= 3.79$$

Solution 15.5

According to Solution 15.1 the difference is:

$$C - P = e^{-rt}\times\left(S_0 e^{(r-\delta)t} - K\right) = e^{-0.04}\left(104.0811 - 100\right) = 3.92$$
$$\underbrace{7.71}_{\text{Solution 15.3}} - \underbrace{3.79}_{\text{Solution 15.4}} = 3.92$$

Solution 15.6

The inversion method results in a simulated value as follows:

$$S_t = S_0\, e^{(r-\delta-0.5\sigma^2)t} \times e^{\sigma\sqrt{t}\,\Phi^{-1}(u)}$$
$$= 100\, e^{(0.08-0.02-0.5\times 0.04)\times 2} \times e^{0.2\times\sqrt{2}\times\Phi^{-1}(0.1401)}$$
$$= 100\, e^{0.08}\times e^{-0.3055} \text{ (since } \Phi^{-1}(0.1401) = -1.08)$$
$$= 79.81$$

Solution 15.7

The simulated value of $S_{0.50}$ is:

$$
\begin{aligned}
S_t &= S_0\, e^{(r-\delta-0.5\sigma^2)t} \times e^{\sigma\sqrt{t}\,\Phi^{-1}(u)} \\
&= 100\, e^{(0.08-0.02-0.5\times0.04)\times0.50} \times e^{0.20\times\sqrt{0.5}\times\Phi^{-1}(0.7054)} \\
&= 100\, e^{0.02} \times e^{0.0764} = 110.12
\end{aligned}
$$

The present value of the call option payoff is:

$$e^{-rt}\times(S_t-K)_+ = e^{-0.08\times0.50}\times(S_{0.50}-102)_+ = 7.80$$

Solution 15.8

Using results of Solution 15.7, the present value of the put option payoff is:

$$e^{-rt}\times(K-S_t)_+ = e^{-0.08\times0.50}\times(102-110.12)_+ = 0 \quad \text{(Solution 15.7: } S_t = 110.12\text{)}$$

Solution 15.9

The starting value and recursion formula for the stock price path are:

$$
\begin{aligned}
&S_0 = 50\ ,\quad h = (1/3)/4 = 1/12 = 0.0833 \\
&S_{hi} = S_{h(i-1)}\, e^{(r-\delta-0.5\sigma^2)h} \times e^{\sigma\sqrt{h}\, z_i} \\
&\quad = S_{h(i-1)} \times 1.00501 \times e^{0.05774 z_i} \quad \text{where } z_i = \Phi^{-1}(u_i)
\end{aligned}
$$

From the standard normal probability table, we have the following path of simulated stock prices:

$$
\begin{aligned}
&\Phi^{-1}(0.9192) = 1.40 \Rightarrow S_h = 50\times1.00501\times e^{0.05774\times1.40} = 54.4810 \\
&\Phi^{-1}(0.4840) = -0.04 \Rightarrow S_{2h} = 54.4810\times1.00501\times e^{0.05774\times(-0.04)} = 54.6278 \\
&\Phi^{-1}(0.6141) = 0.29 \Rightarrow S_{3h} = 54.6278\times1.00501\times e^{0.05774\times0.29} = 55.8286 \\
&\Phi^{-1}(0.2420) = -0.70 \Rightarrow S_{4h} = 55.8286\times1.00501\times e^{0.05774\times(-0.70)} = 53.8860
\end{aligned}
$$

$$\Rightarrow \bar{S}_t = \frac{S_h + S_{2h} + S_{3h} + S_{4h}}{4} = 54.7059$$

$$\Rightarrow Y = e^{-rt}\left(\bar{S}_t - K\right)_+ = e^{-0.08\times(1/3)}\times(54.7059-52)_+ = 2.63$$

Solution 15.10

$$z_1 = \Phi^{-1}(0.5793) = 0.20$$
$$z_2 = \Phi^{-1}(0.3300) = -0.44$$

$$z_S = z_1 = 0.20$$
$$z_T = \rho z_1 + \sqrt{1-\rho^2}\, z_2 \;= 0.20\times 0.20 + \sqrt{0.96}\times(-0.44) = -0.3911$$

$$R_S(0,t) = \left(\alpha_S - \delta_S - 0.5\sigma_S^2\right)t + \sigma_S\sqrt{t}\; z_S =$$
$$= (0.07 - 0.5\times 0.16)\times 1 \;+ 0.4\sqrt{1}\times 0.20 = 0.07$$
$$\Rightarrow \quad S_t = S_0\, e^{R_S(0,t)} = 100\, e^{0.07} = 107.25$$

$$R_T(0,t) = \left(\alpha_T - \delta_T - 0.5\sigma_T^2\right)t + \sigma_T\sqrt{t}\; z_T$$
$$= (0.06 - 0.5\times 0.04)\times 1 \;+ 0.2\sqrt{1}\times(-0.3911) =$$
$$\Rightarrow \quad T_t = T_0\, e^{R_T(0,t)} = 90\; e^{-0.0382} = 86.62$$

Solution 15.11

1. $z = \Phi^{-1}(u_1) = \Phi^{-1}(0.6772) = 0.46$

2. Use the random number $u_2 =$ 0.8238 to simulate $n = N(1) =$ Poisson mean 0.10

$\Pr\left(N(1)=0\right) = e^{-0.20} = 0.81873$

$\Pr\left(N(1) \le 1\right) = e^{-0.20}(1+0.20) = 0.98248$

$0.81873 < u_2 < 0.98248 \;\Rightarrow\; n = 1$

3. Use random number u_3 to simulate $w_1 = \Phi^{-1}(u_3) = \Phi^{-1}(0.3446) = -0.40$

4. The simulated stock price at time 1 is thus:

$k = e^{\alpha_J} - 1 = -0.07688$

$$S_t = S_0\, e^{(\alpha-\delta-\lambda k - 0.5\sigma^2)t + \sigma\sqrt{t}\,z \;+ (\alpha_J - 0.5\sigma_J^2)n + \sigma_J(w_1+\cdots+w_n)}$$
$$= 100\, e^{(0.09 - 0 + 0.20\times 0.07688 - 0.5\times 0.09) + 0.3\times 0.46 + (-0.08 - 0.5\times 0.0016) + 0.3\times(-0.40)}$$
$$= 100\, e^{-0.0024} = 99.76$$

Solution 15.12

$$Y_t = e^{(\alpha - 0.5\sigma^2)t + \sigma\sqrt{t}Z} \times e^{\beta N(t)}$$

$$= e^{(\alpha - 0.5\sigma^2)t} \times e^{N(0,\sigma^2 t)} \times e^{\beta N(t)}$$

$$\Rightarrow E[Y_t] = e^{(\alpha - 0.5\sigma^2)t} \times e^{0.5\sigma^2 t} \times M_{N(t)}(\beta)$$

$$= e^{\alpha t} \times e^{\lambda t(e^{\beta}-1)} = e^{\alpha t} \times e^{\lambda t k}$$

$$= e^{(\alpha+\lambda k)t}$$

$$Y_t^2 = e^{(\alpha - 0.5\sigma^2)2t + 2\sigma\sqrt{t}Z} \times e^{2\beta N(t)}$$

$$= e^{(\alpha - 0.5\sigma^2)2t} \times e^{N(0,4\sigma^2 t)} \times e^{2\beta N(t)}$$

$$\Rightarrow E\left[Y_t^2\right] = e^{(\alpha - 0.5\sigma^2)2t} \times e^{2\sigma^2 t} \times M_{N(t)}(2\beta)$$

$$= e^{(2\alpha+\sigma^2)t} \times e^{\lambda t(e^{2\beta}-1)} = e^{(2\alpha+\sigma^2)t} \times e^{\lambda t(k^2+2k)}$$

$$= e^{(2\alpha+\sigma^2+2\lambda k+\lambda k^2)t}$$

$$\Rightarrow \operatorname{var}(Y_t) = e^{(2\alpha+\sigma^2+2\lambda k+\lambda k^2)t} - \left(e^{(\alpha+\lambda k)t}\right)^2$$

$$= e^{(2\alpha+2\lambda k)t}\left(e^{(\sigma^2+\lambda k^2)t} - 1\right)$$

