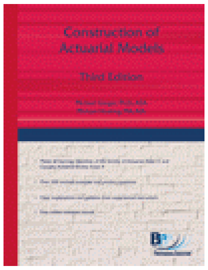

BPP PROFESSIONAL EDUCATION

Construction of Actuarial Models

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Solutions to practice questions – Chapter 14

Solution 14.1

$$\ln\left(X_1 \times \cdots \times X_n\right) = \sum_{i=1}^{n} \ln\left(X_i\right) \sim N\left(\sum_{i=1}^{n} \mu_i \,,\, \sum_{i=1}^{n} \sigma_i^2\right)$$

$$\Rightarrow X_1 \times \cdots \times X_n \sim \text{lognormal with parameters } \mu = \sum_{i=1}^{n} \mu_i \,,\ \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

Solution 14.2

1. $e^{\mu + 0.5\sigma^2} = E[X] = 40.44730 \Rightarrow \mu + 0.5\sigma^2 = 3.700$
2. $e^{2\mu + 2\sigma^2} = E\left[X^2\right] = \left(E[X]\right)^2 + \text{var}(X) = 2{,}440.60163 \Rightarrow 2\mu + 2\sigma^2 = 7.800$
3. $\sigma^2 = 7.800 - 2 \times 3.700 = 0.400 \,,\ \mu = 3.500$

Solution 14.3

1. $z_{0.10} = 1.282 = 90^{\text{th}}$ percentile of $N(0,1)$
2. $\mu + 1.282\sigma = 3.5 + 1.282 \times \sqrt{0.4} = 4.31081 = 90^{\text{th}}$ percentile of $N\left(\mu, \sigma^2\right)$
3. $e^{\mu + 1.282\sigma} = e^{4.31081} = 74.50 = 90^{\text{th}}$ percentile of $e^{N\left(\mu, \sigma^2\right)}$

Solution 14.4

$$\Pr\left(30 \le e^{N(\mu,\sigma^2)} \le 50\right) = \Pr\left(\ln(30) \le N(\mu,\sigma^2) \le \ln(50)\right)$$
$$= \Phi\left(\frac{\ln(50)-\mu}{\sigma}\right) - \Phi\left(\frac{\ln(30)-\mu}{\sigma}\right)$$
$$= \Phi(0.651) - \Phi(-0.156)$$
$$= 0.7426 - 0.4379 = 0.3047$$

Solution 14.5

$$l = \frac{\ln(x)-\mu-\sigma^2}{\sigma} = \frac{\ln(38) - 3.5 - 0.40}{\sqrt{0.40}} = -0.415 \Rightarrow \Phi(l) = 0.3391$$
$$u = \frac{\ln(x)-\mu}{\sigma} = \frac{\ln(38) - 3.5}{\sqrt{0.40}} = 0.218 \Rightarrow \Phi(u) = 0.5863$$
$$E[X \mid X < x] = e^{\mu+0.5\sigma^2} \times \frac{\Phi(l)}{\Phi(u)} = e^{3.7} \times \frac{0.3391}{0.5861} = 23.40$$

Solution 14.6

$$l = \frac{\ln(x)-\mu-\sigma^2}{\sigma} = \frac{\ln 45 - 3.5 - 0.40}{\sqrt{0.40}} = -0.148 \Rightarrow 1 - \Phi(l) = 0.5587$$
$$u = \frac{\ln(x)-\mu}{\sigma} = \frac{\ln(45) - 3.5}{\sqrt{0.40}} = 0.485 \Rightarrow \Phi(u) = 0.6861$$
$$E[X \mid X > x] = e^{\mu+0.5\sigma^2} \times \frac{1-\Phi(l)}{1-\Phi(u)} = e^{3.7} \times \frac{0.5588}{0.3138} = 71.99$$

Solution 14.7

$$104.08 = E[S_t] = S_0\, e^{(\alpha-\delta)t} = 100e^{(\alpha-\delta)t} \Rightarrow e^{(\alpha-\delta)t} = 1.0408$$
$$E[S_{2.5t}] = S_0\, e^{(\alpha-\delta)2.5t} = 100\left(e^{(\alpha-\delta)t}\right)^{2.5} = 110.51$$

Solution 14.8

$$\alpha = 0.10\ ,\ \delta = 0\ ,\ \sigma = 0.20\ ,\ S_0 = 100$$
$$E[S_t] = S_0\, e^{(\alpha-\delta)t} \Rightarrow E[S_2] = 100 \times e^{0.20} = 122.14$$
$$\operatorname{var}(S_t) = S_0^2\, e^{2(\alpha-\delta)t}\left(e^{\sigma^2 t} - 1\right) \Rightarrow \operatorname{var}(S_2) = 100^2 \times e^{0.4} \times \left(e^{0.08} - 1\right) = 1{,}242.50$$

Solution 14.9

Since returns on intervals of equal length are identically distributed, we have:

$$\frac{S_2}{S_1} = \frac{S_0 e^{R(0,2)}}{S_0 e^{R(0,1)}} = \frac{e^{R(0,1)+R(1,2)}}{e^{R(0,1)}} = e^{R(1,2)} \sim e^{R(0,1)}$$

So the distribution of $S_2 \mid S_1 = 105$ is the same as the distribution of S_1 when $S_0 = 105$:

$$\alpha = 0.10\ ,\ \delta = 0\ ,\ \sigma = 0.20\ ,\ S_0 = 105$$
$$E[S_t] = S_0\, e^{(\alpha-\delta)t} \Rightarrow E[S_1] = 105 \times e^{0.10} = 116.04$$
$$\text{var}(S_t) = S_0^2\, e^{2(\alpha-\delta)t}\left(e^{\sigma^2 t} - 1\right) \Rightarrow \text{var}(S_1) = 105^2 \times e^{0.2} \times \left(e^{0.04} - 1\right) = 549.56$$

Solution 14.10

$$\alpha = 0.10\ ,\ \delta = 0\ ,\ \sigma = 0.20\ ,\ S_0 = 100$$
$$\text{median of } S_t = S_0 e^{(\alpha-\delta-0.5\sigma^2)t} \Rightarrow$$
$$\text{median of } S_1 = S_0 e^{(\alpha-\delta-0.5\sigma^2)} = 100 \times e^{0.08} = 108.33$$

Solution 14.11

$\alpha = 0.10$, $\delta = 0$, $\sigma = 0.20$, $S_0 = 100$, $K = 100$, $t = 0.5$

1. $E[S_{0.5}] = S_0\, e^{(\alpha-\delta)0.5} = 100 \times e^{0.05} = 105.127$

2. $$l = \frac{\ln(K/S_0) - (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(100/100) - 0.12 \times 0.5}{0.2 \times \sqrt{0.5}} = -0.424$$

   $\Rightarrow \Phi(l) = 0.3358$

   $$u = \frac{\ln(K/S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \frac{\ln(100/100) - 0.08 \times 0.5}{0.2 \times \sqrt{0.5}} = -0.283$$

   $\Rightarrow \Phi(u) = 0.3886$

3. $$E[S_t \mid S_t < K] = E[S_t] \times \frac{\Phi(l)}{\Phi(u)}$$

   $\Rightarrow E[S_{0.5} \mid S_{0.5} < 100] = 90.84$

Solution 14.12

Using the results of Solution 14.11, we have:

$$E\left[(K - S_t)_+\right] = K\Phi(u) - E[S_t]\,\Phi(l) = 100 \times 0.3886 - 105.127 \times 0.3358 = 3.575$$

Solution 14.13

Using the results of Solution 14.11, we have:

$$K - E\left[S_t \mid S_t < K\right] = K - E\left[S_t\right] \times \frac{\Phi(l)}{\Phi(u)} = 100 - 105.127 \times \frac{0.3358}{0.3886} = 9.20$$

Solution 14.14

$\alpha = 0.10$, $\delta = 0$, $\sigma = 0.20$, $S_0 = 100$, $K = 110$, $t = 0.5$

1. $E\left[S_{0.5}\right] = S_0\, e^{(\alpha-\delta)0.5} = 100 \times e^{0.05} = 105.127$

2. $l = \dfrac{\ln\left(K/S_0\right) - (\alpha - \delta + 0.5\sigma^2)t}{\sigma\sqrt{t}} = \dfrac{\ln\left(110/100\right) - 0.12 \times 0.5}{0.2 \times \sqrt{0.5}} = 0.250$

   $\Rightarrow \Phi(l) = 0.5986$

   $u = \dfrac{\ln\left(K/S_0\right) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} = \dfrac{\ln\left(110/100\right) - 0.08 \times 0.5}{0.2 \times \sqrt{0.5}} = 0.391$

   $\Rightarrow \Phi(u) = 0.6521$

3. $E\left[S_t \mid S_t > K\right] = E\left[S_t\right] \times \dfrac{1 - \Phi(l)}{1 - \Phi(u)}$

   $\Rightarrow E\left[S_{0.5} \mid S_{0.5} > 110\right] = 121.31$

Solution 14.15

Using the results of Solution 14.14, we have:

$$E\left[\left(S_t - K\right)_+\right] = E\left[S_t\right]\Phi(-l) - K\Phi(-u)$$
$$= 105.127 \times \left(1 - 0.5987\right) - 110 \times \left(1 - 0.6521\right) = 3.935$$

Solution 14.16

Using the results of Solution 14.14, we have:

$$E\left[S_t \mid S_t > K\right] - K = E\left[S_t\right] \times \frac{\Phi(-l)}{\Phi(-u)} - K$$
$$= 105.127 \times \frac{1 - 0.5987}{1 - 0.6521} - 110 = 11.26$$

Solution 14.17

$$\text{call payoff} = (S_t - K)_+ = \begin{cases} 0 & \text{if } S_t \le K \\ S_t - K & \text{if } S_t > K \end{cases}$$

$$\text{put payoff} = (K - S_t)_+ = \begin{cases} K - S_t & \text{if } S_t \le K \\ 0 & \text{if } S_t > K \end{cases}$$

$$\Rightarrow \text{ call payoff} - \text{put payoff} = S_t - K$$

Solution 14.18

Using the results of Solutions 14.12 and 14.17, we have:

$$\text{Solution 14.17}: \; E[\text{call payoff}] - \underbrace{E[\text{put payoff}]}_{\substack{= 3.575 \\ \text{(see Solution 14.12)}}} = \underbrace{E[S_t]}_{105.127} - \underbrace{K}_{100}$$

$$\Rightarrow E[\text{call payoff}] = 8.702$$

Solution 14.19

From the given weekly price data we can determine the corresponding weekly returns:

weekly prices: 100, 102, 106, 98, 100, 97, 100, 103, 108, 102 $\Rightarrow$

$r_1 = \ln(102/100) = 0.0198$ $\quad$ $r_2 = \ln(106/102) = 0.0385$

$r_3 = \ln(98/106) = -0.0785$ $\quad$ $r_4 = \ln(100/98) = 0.0202$

$r_5 = \ln(97/100) = -0.0305$ $\quad$ $r_6 = \ln(100/97) = 0.0305$

$r_7 = \ln(103/100) = 0.0296$ $\quad$ $r_8 = \ln(108/103) = 0.0474$

$r_9 = \ln(102/108) = -0.0572$

Now determine the estimates:

$h = 1/52$ (week is a 52^{nd} of a year)

$$\bar{r} = \frac{1}{n}(r_1 + \cdots + r_n) = \frac{1}{9}(0.0198) = 0.0022$$

$$\hat{\sigma}^2 = \frac{1}{h \times (n-1)} \sum_{i=1}^{n} (r_i - \bar{r})^2 = \frac{1}{h \times (n-1)} \left(\sum_{i=1}^{n} r_i^2 - n\bar{r}^2 \right) = \frac{1}{(1/52) \times 8} \left(0.01668 - 9 \times 0.0022^2 \right) = 0.10814$$

$$\Rightarrow \hat{\sigma} = 0.3288$$

Solution 14.20

$$\hat{\alpha} = \frac{\bar{r}}{h} + 0.5\hat{\sigma}^2 = 0.1685 \quad \text{(See Solution 14.19)}$$