Construction of Actuarial Models
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Solutions to practice questions – Chapter 11

Solution 11.1

From the assumption of a conditional gamma distribution, along with standard gamma moment formulas, we have:

$$\mu(\Theta) = E[X|\Theta] = 2\Theta \quad , \quad v(\Theta) = \text{var}(X|\Theta) = 2\Theta^2$$

From the assumption that Θ is uniformly distributed on $[50,100]$, along with standard uniform moment formulas, we have:

$$\mu = E[X] = E[\mu(\Theta)] = E[2\Theta] = 2 \times 75 = 150$$

$$v = E[\text{var}(X|\Theta)] = E[2\Theta^2] = 2\left(75^2 + \frac{50^2}{12}\right) = 11,666.67$$

$$a = \text{var}(\mu(\Theta)) = \text{var}(2\Theta) = 4 \times \frac{50^2}{12} = 833.33$$

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Solution 11.2

From the assumption of a conditional gamma distribution, along with standard gamma moment formulas, we have:

$$\mu(\Theta) = E[X|\Theta] = 2\Theta \quad , \quad v(\Theta) = \text{var}(X|\Theta) = 2\Theta^2$$

From the assumption that Θ is either 25 or 100 with respective probabilities 0.6 and 0.4, we have:

$$E[\Theta] = 55 \quad , \quad E[\Theta^2] = 4,375$$

The remaining structural parameters are:

$$\mu = E[X] = E[\mu(\Theta)] = E[2\Theta] = 2 \times 55 = 110$$

$$v = E[\text{var}(X|\Theta)] = E[2\Theta^2] = 2 \times 4,375 = 8,750$$

$$a = \text{var}(\mu(\Theta)) = \text{var}(2\Theta) = 4 \times (4,375 - 55^2) = 5,400$$

Solution 11.3

From the given data we have $\bar{x}_3 = (125 + 240 + 90)/3 = 151.67$

From the results in Solution 11.1, we have the following credibility prediction:

$$Z = \frac{n}{n + (v/a)} = \frac{3}{3 + 14} = \frac{3}{17}$$

$$Z\bar{x}_3 + (1-Z)E[X] = \frac{3}{17} \times 151.67 + \frac{14}{17} \times 150 = 150.29$$

Solution 11.4

From the given data we have $\bar{x}_3 = (125 + 240 + 90)/3 = 151.67$

From the results in Solution 11.2, we have the following credibility prediction:

$$Z = \frac{n}{n + (v/a)} = \frac{3}{3 + 1.62037} = 0.64930$$

$$Z\bar{x}_3 + (1-Z)E[X] = 0.64930 \times 151.67 + 0.35070 \times 110 = 137.05$$

Solution 11.5

$$\frac{2}{3} = Z_{100} = \frac{100}{100 + (v/a)} \Rightarrow v/a = 50$$

$$\Rightarrow Z_{200} = \frac{200}{200 + (v/a)} = \frac{200}{250} = 0.80$$

Solution 11.6

In Solution 11.5 we saw that $v/a = 50$. So to attain 90% credibility we must have:

$$0.90 = Z = \frac{n}{n + (v/a)} = \frac{n}{n + 50} \Rightarrow n = 450$$

Solution 11.7

Since they have the same risk parameter the covariance between N_1 and N_2 is the same as Bühlmann's a, the variance of the hypothetical means (see Section 11.2). From the conditional negative binomial assumptions and standard moment formulas, we have:

$$\mu(\Theta) = E[N|\Theta] = 2\Theta$$

Since Θ is assumed to be uniform on $[5,10]$, we have:

$$\text{var}(\Theta) = \frac{5^2}{12} = \frac{25}{12}$$

Finally, we have:

$$\text{cov}(N_1, N_2) = a = \text{var}(\mu(\Theta)) = \text{var}(2\Theta) = 4 \times \frac{25}{12} = \frac{25}{3}$$

Solution 11.8

According to a result in Section 11.2 , $\text{var}(\bar{N}_n) = \frac{v}{n} + a$. Here we have $n = 2$ observations. In Solution 11.7 we saw that $a = 25/3$. So we must compute v:

$$v(\Theta) = \text{var}(N \mid \Theta) = 2\Theta(1 + \Theta)$$

$$\Rightarrow v = E[v(\Theta)] = E[2\Theta(1 + \Theta)] = 2 \times \left(E[\Theta] + E[\Theta^2]\right)$$

$$= 2 \times \left(7.5 + \left(7.5^2 + \frac{5^2}{12}\right)\right) = 131.67$$

$$\Rightarrow \text{var}(\bar{N}_2) = \frac{v}{2} + a = 65.83 + \frac{25}{3} = 74.17$$

Solution 11.9

According to a result in Section 11.2, the ESE is equal to $v + (1 - Z)a$. From Solutions 11.1 and 11.3 we have:

$$v = 11{,}666.67 \ , \quad a = 833.33 \ , \quad Z = \frac{3}{17}$$

$$\Rightarrow ESE = v + (1 - Z)a = 12{,}352.94$$

Solution 11.10

$$\mu(\Lambda) = E[N \mid \Lambda] = \Lambda \quad \text{(Poisson formula)}$$

$$v(\Lambda) = \text{var}(N \mid \Lambda) = \Lambda \quad \text{(Poisson formula)}$$

$$\mu = E[N] = E[\mu(\Lambda)] = E[\Lambda] = \alpha\theta = 2 \times 0.15 = 0.30 \quad \text{(gamma formula)}$$

$$v = E[v(\Lambda)] = E[\Lambda] = 0.30 \quad \text{(same as above)}$$

$$a = \text{var}(\mu(\Lambda)) = \text{var}(\Lambda) = \alpha\theta^2 = 2 \times 0.15^2 = 0.045 \quad \text{(gamma formula)}$$

$$Z = \frac{n}{n + (v/a)} = \frac{4}{4 + (0.30/0.045)} = 0.375$$

$$\text{prediction} = Z\bar{n}_4 + (1 - Z)E[N] = 0.375 \times \frac{0}{4} + 0.625 \times 0.30 = 0.1875$$

Note: This is the Poisson/gamma model where the Bayesian credibility estimate is linear. So you could also solve this problem from memorized Bayesian results for this model. See Chapter 10.

Solution 11.11

From (i) and (ii) we have:

$$v(\Lambda) = \text{var}(N \mid \Lambda) = \Lambda \quad \text{(Poisson formula)}$$
$$v = E[v(\Lambda)] = E[\Lambda] = 0.30$$

There is not enough information to compute $a = \text{var}(E[N|\Lambda]) = \text{var}(\Lambda)$ directly. However, we can compute $\text{var}(N)$ from (iii), and we know that $\text{var}(N) = v + a$:

$$\text{(iii)} \Rightarrow \text{var}(N) = r\beta(1+\beta) = 2 \times 0.15 \times 1.15 = 0.345$$
$$0.345 = \text{var}(N) = v + a = 0.30 + a \Rightarrow a = 0.045$$
$$\Rightarrow k = v / a = 0.30 / 0.045 = 6.667 \Rightarrow Z = \frac{3}{3 + 6.667} = 0.3103$$

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Solution 11.12

Note first that the distribution of Θ is beta with parameters $a = 3, b = 1$. So from beta moment formulas we have:

$$E[\Theta] = \frac{a}{a+b} = \frac{3}{4} \; , \; E[\Theta^2] = \frac{a}{a+b} \times \frac{a+1}{a+b+1} = \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}$$

The structural parameters are:

$$\mu(\Theta) = E[N \mid \Theta] = 3\Theta \quad \text{(binomial formula)}$$
$$v(\Theta) = \text{var}(N \mid \Theta) = 3\Theta(1-\Theta) \quad \text{(binomial formula)}$$
$$\mu = E[N] = E[\mu(\Theta)] = E[3\Theta] = \frac{9}{4} \quad \text{(beta formula)}$$
$$v = E[v(\Theta)] = E[3\Theta(1-\Theta)] = 3 \times \left(\frac{3}{4} - \frac{3}{5}\right) = 0.45 \quad \text{(beta formulas)}$$
$$a = \text{var}(\mu(\Theta)) = \text{var}(3\Theta) = 9\,\text{var}(\Theta) = 9\left(\frac{3}{5} - \left(\frac{3}{4}\right)^2\right) = 0.3375 \quad \text{(beta formula)}$$

$$Z = \frac{n}{n + (v/a)} = \frac{2}{2 + (0.45 / 0.33750)} = 0.6$$
$$\text{prediction} = Z\bar{n}_2 + (1-Z)E[N] = 0.60 \times \frac{6}{2} + 0.40 \times \frac{9}{4} = 2.7$$

Note: This is the binomial/beta model where the Bayesian credibility estimate is linear. So you could also solve this problem from memorized Bayesian results for this model. See Chapter 10.

Solution 11.13

$$\mu(\Lambda) = E[X \mid \Lambda] = \Lambda \quad \text{(exponential formula)}$$

$$v(\Lambda) = \mathrm{var}(X \mid \Lambda) = \Lambda^2 \quad \text{(exponential formula)}$$

$$\mu = E[X] = E[\mu(\Lambda)] = E[\Lambda] = \frac{\theta}{\alpha-1} = \frac{400}{2} = 200 \quad \text{(Pareto formula)}$$

$$v = E[v(\Lambda)] = E[\Lambda^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{2 \times 400^2}{2} = 160,000 \quad \text{(Pareto formula)}$$

$$a = \mathrm{var}(\mu(\Lambda)) = \mathrm{var}(\Lambda) = 160,000 - 200^2 = 120,000 \quad \text{(Pareto formula)}$$

$$Z = \frac{n}{n + (v/a)} = \frac{4}{4 + (4/3)} = 0.75$$

$$\text{prediction} = Z\bar{x}_4 + (1-Z)E[X] = 0.75 \times \frac{1,210}{4} + 0.25 \times 200 = 276.88$$

Solution 11.14

From (ii) we can see that the distribution of the risk parameter Λ is a two-point mixture of two gamma distributions:

　　30%　gamma with parameters $\alpha = 2, \theta = 1/7$
　　70%　gamma with parameters $\alpha = 2, \theta = 1/3$

For a gamma distribution, we have:

　　first moment $= \alpha\theta$ ，　second moment $= \alpha(\alpha+1)\theta^2$

So the first two moments for our 2-point mixture are:

$$E[\Lambda] = 0.3 \times \left(2 \times \frac{1}{7}\right) + 0.7 \times \left(2 \times \frac{1}{3}\right) = 0.55238$$

$$E[\Lambda^2] = 0.3 \times \left(2 \times 3 \times \left(\frac{1}{7}\right)^2\right) + 0.7 \times \left(2 \times 3 \times \left(\frac{1}{3}\right)^2\right) = 0.50340$$

The structural parameters are:

$$\mu(\Lambda) = E[X \mid \Lambda] = \Lambda \quad \text{(Poisson formula)}$$

$$v(\Lambda) = \mathrm{var}(X \mid \Lambda) = \Lambda \quad \text{(Poisson formula)}$$

$$\mu = E[X] = E[\mu(\Lambda)] = E[\Lambda] = 0.55238 \quad \text{(2-point gamma mix)}$$

$$v = E[v(\Lambda)] = E[\Lambda] = 0.55238 \quad \text{(2-point gamma mix)}$$

$$a = \mathrm{var}(\mu(\Lambda)) = \mathrm{var}(\Lambda) = 0.50340 - 0.55238^2 = 0.19828 \quad \text{(2-point gamma mix)}$$

$$Z = \frac{n}{n + (v/a)} = \frac{2}{2 + 2.78591} = 0.41789$$

$$\text{prediction} = Z\bar{x}_2 + (1-Z)E[X] = 0.41789 \times \frac{0}{2} + 0.58211 \times 0.55238 = 0.32154$$

Solution 11.15

In Section 11.1 we indicated that by substituting $m = 1$ into the formulas for the averages Y_i we could obtain formulas for the hidden underlying $X's$, thus turning a Bühlmann-Straub problem into an ordinary Bühlmann problem. Substituting $m = 1$ results in our old friend from Chapter 10, the normal/normal model, where the Bayesian prediction is a linear function of the data. Recall that this means the Bayesian and Bühlmann predictions are identical. So let's make this one quick and use a Chapter 10 formula for the Bayesian credibility prediction.

For the normal/normal model, we are given:

$$\mu = 200 \,,\, \sigma_1^2 = 275 \,,\, \sigma_2^2 = 625$$

$$n = 20 + 10 = 30$$

$$\bar{x}_{30} = \frac{20 \times 225 + 10 \times 210}{30} = 220$$

$$\mu^* = \left(\frac{\sum x_i}{\sigma_1^2} + \frac{\mu}{\sigma_2^2} \right) \Big/ \left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) = \left(\frac{30 \times 220}{275} + \frac{200}{625} \right) \Big/ \left(\frac{30}{275} + \frac{1}{625} \right) = 219.71$$

linerar Bayesian $\Rightarrow$ Bühlmann $\Rightarrow$ Bayesian $= \mu^*$ (posterior mean) $= 219.71$

Solution 11.16

From the first two columns of the joint probability table we can determine the conditional distributions of X for $\theta = 0$ and $\theta = 1$:

X	$\Pr(X = x \mid \theta = 0)$	$\Pr(X = x \mid \theta = 1)$
0	3/7	3/6
1	2/7	2/6
2	2/7	1/6

From this table we can construct a table of values of the conditional means and variances:

Θ	$\Pr(\Theta = \theta)$	$E[X\|\Theta]$	$\text{var}(X \mid \Theta)$
0	0.7	6/7	34/49
1	0.3	4/6	20/36

The marginal probabilities for the risk parameter in the table above are obtained by summing down the columns in the joint probability table.

Solution 11.17

From the results in Solution 11.16, we have:

$$\mu = E[X] = E\big[E[X|\Theta]\big] = 0.7 \times \frac{6}{7} + 0.3 \times \frac{4}{6} = 0.8$$

$$v = E\big[\,\mathrm{var}(X|\Theta)\big] = 0.7 \times \frac{34}{49} + 0.3 \times \frac{20}{36} = 0.65238$$

$$a = \mathrm{var}\big(E[X|\Theta]\big) = \left(0.7 \times \left(\frac{6}{7}\right)^2 + 0.3 \times \left(\frac{4}{6}\right)^2\right) - 0.80^2 = 0.00762$$

$$Z = \frac{n}{n + (v/a)} = \frac{10}{10 + 85.625} = 0.10458$$

Note: We could have obtained the marginal distribution of X by summing across the rows of the joint probability table. This would provide an alternative way to determine the overall mean $E[X]$ and a way to check the computations of v and a since $\mathrm{var}(X) = v + a$.

Solution 11.18

From the results of Solutions 11.16 and 11.17, we have:

$$\text{prediction} = Z\bar{x}_{10} + (1 - Z)E[X] = 0.10458 \times \frac{4}{10} + 0.89542 \times 0.8 = 0.75817$$

Solution 11.19

From the conditional PDF for the annual claim amount, we have:

$$f_X\left(x \mid \Theta = \theta\right) = \frac{2\left(\theta - x\right)}{\theta^2} \quad \text{for } 0 \le x \le \theta$$

$$\Rightarrow \quad E\left[X \mid \Theta\right] = \int_0^\Theta x \times \frac{2\left(\Theta - x\right)}{\Theta^2}\, dx = \frac{\Theta}{3}$$

$$E\left[X^2 \mid \Theta\right] = \int_0^\Theta x^2 \times \frac{2\left(\Theta - x\right)}{\Theta^2}\, dx = \frac{\Theta^2}{6}$$

$$\mathrm{var}\left(X \mid \Theta\right) = \frac{\Theta^2}{18}$$

From the distribution of Θ and the results above, we have:

$$\pi\left(\theta\right) = \frac{2\theta}{7500} \quad \text{for } 50 \le \theta \le 100$$

$$\Rightarrow \quad \mu = E[X] = E\left[E\left[X \mid \Theta\right]\right] = E\left[\frac{\Theta}{3}\right] = \frac{1}{3} \times \int_{50}^{100} \theta \times \frac{2\theta}{7500}\, d\theta = 25.92593 \quad \left(\Rightarrow E[\Theta] = 77.77778\right)$$

$$v = E\left[\mathrm{var}\left(X \mid \Theta\right)\right] = E\left[\frac{\Theta^2}{18}\right] = \frac{1}{18} \times \int_{50}^{100} \theta^2 \times \frac{2\theta}{7500}\, d\theta = 347.22222 \quad \left(\Rightarrow E\left[\Theta^2\right] = 6,250\right)$$

$$a = \mathrm{var}\left(E\left[X \mid \Theta\right]\right) = \mathrm{var}\left(\frac{\Theta}{3}\right) = \frac{\mathrm{var}\left(\Theta\right)}{9} = \frac{6,250 - \left(77.77778\right)^2}{9} = 22.29081$$

Solution 11.20

From the results of Solution 11.19, we have:

$$Z = \frac{1}{1 + \left(v / a\right)} = \frac{1}{1 + \left(15.57692\right)} = 0.06032$$

$$\text{prediction} = Z\bar{x}_1 + \left(1 - Z\right)E[X] = 0.06032 \times 50 + 0.93968 \times 25.92593 = 27.38$$