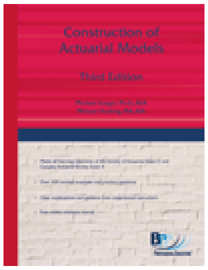

Construction of Actuarial Models

Third Edition

by Mike Gauger and Michael Hosking

Published by BPP Professional Education

Solutions to practice questions – Chapter 10

Solution 10.1

Hypothetical mean: $E\left[X|\Theta\right] = \sum_x x\Pr\left(X = x|\Theta\right) =$

$$= 0 \times 2\Theta \ + \ 1\times\Theta + 2\times(1-3\Theta) = 2 - 5\Theta$$

Pure premium: $E[X] \ = \ E\left[E\left[X|\Theta\right]\right] = E[2-5\Theta] = 2 - 5 \times \dfrac{1/3}{2}$

$$= 7/6$$

Solution 10.2

$$\text{var}\left[E(X\,|\,\theta)\right] = \text{var}[2-5\theta] = 25\,\text{var}(\theta)$$

But the variance of a uniform distribution is:

$$\frac{(b-a)^2}{12} = \frac{(1/3)^2}{12} = \frac{1}{108}$$

So: $\text{cov}[X_1, X_2] = \text{var}\left[E(X\,|\,\theta)\right] = \dfrac{25}{108}$

Solution 10.3

This is asking for the Bayesian credibility premium $E\left[X_2\,|\,X_1\right]$.

Solution 10.4

Step 1. Determine $\Pr(X_1 = 0,\, X_2 = x_2)$

$$\Pr(X_1 = 0,\, X_2 = 0) = \int_0^{1/3} f_{X_1,X_2,\Theta}(0,0,\theta)\,d\theta = \int_0^{1/3} f_\Theta(\theta)\, f_X(0|\theta)\, f_X(0\,|\,\theta)\,d\theta$$
$$= \int_0^{1/3} 3\times(2\theta)\times(2\theta)\,d\theta = \frac{4}{27}$$
$$\Pr(X_1 = 0,\, X_2 = 1) = \int_0^{1/3} f_{X_1,X_2,\Theta}(0,1,\theta)\,d\theta = \int_0^{1/3} f_\Theta(\theta)\, f_X(0|\theta)\, f_X(1\,|\,\theta)\,d\theta$$
$$= \int_0^{1/3} 3\times(2\theta)\times(\theta)\,d\theta = \frac{2}{27}$$
$$\Pr(X_1 = 0,\, X_2 = 2) = \int_0^{1/3} f_{X_1,X_2,\Theta}(0,0,\theta)\,d\theta = \int_0^{1/3} f_\Theta(\theta)\, f_X(0|\theta)\, f_X(2\,|\,\theta)\,d\theta$$
$$= \int_0^{1/3} 3\times(2\theta)\times(1-3\theta)\,d\theta = \frac{3}{27}$$

Step 2. Determine $\Pr(X_1 = 0)$

$$\Pr(X_1 = 0) = \sum_{x_2} \Pr(X_1 = 0, X_2 = x_2) = \frac{4+2+3}{27} = \frac{1}{3}$$

Step 3. Determine $\Pr(X_2 = x_2 | X_1 = 0) = \Pr(X_1 = 0, X_2 = x_2) / \Pr(X_1 = 0)$

$$\Pr(X_2 = 0|X_1 = 0) = \frac{4/27}{1/3} = \frac{4}{9}$$
$$\Pr(X_2 = 1|X_1 = 0) = \frac{2/27}{1/3} = \frac{2}{9}$$
$$\Pr(X_2 = 2|X_1 = 0) = \frac{3/27}{1/3} = \frac{3}{9}$$

Solution 10.5

The Bayesian credibility prediction $E[X_2 | X_1 = 0]$ is the mean of the predictive distribution in Solution 10.4:

$$E[X_2 | X_1 = 0] = \sum_{x_2} x_2 \times \Pr(X_2 = x_2 | X_1 = 0)$$
$$= 0\times\frac{4}{9} + 1\times\frac{2}{9} + 2\times\frac{3}{9} = \frac{8}{9}$$

Solution 10.6

First, apply the double expectation theorem:

$$E[X_2 \,|\, X_1 = 0] = E\big[E[X_2 | \Theta] \,|\, X_1 = 0\big] = E[2 - 5\Theta |\, X_1 = 0] \quad \text{(Solution 10.1)}$$
$$= 2 - 5E[\Theta | X_1 = 0]$$

Now determine the posterior distribution and the posterior mean:

$$f_\Theta(\theta | X_1 = 0) = c\ f_\Theta(\theta)\ \Pr(X_1 = 0 | \theta) = c \times 3 \times 2\theta$$
$$1 = \int_0^{1/3} f_\Theta(\theta | X_1 = 0)\, d\theta = \frac{3c}{9} \Rightarrow c = 3$$
$$\Rightarrow f_\Theta(\theta | X_1 = 0) = 18\theta \quad \text{for } 0 \le \theta \le 1/3$$
$$\Rightarrow E[\Theta | X_1 = 0] = \int_0^{1/3} \theta \times 18\theta\, d\theta = \frac{6}{27}$$

So the credibility estimate is:

$$E[X_2 | X_1 = 0] = 2 - 5E[\Theta | X_1 = 0]$$
$$= 2 - 5 \times \frac{6}{27} = \frac{24}{27} = \frac{8}{9}$$

Solution 10.7

This is the Poisson/gamma model since an exponential with mean 0.25 is a gamma with $\alpha = 1, \theta = 0.25$:

$$x_1 = x_2 = 0 \Rightarrow$$
$$\alpha^* = \alpha + \sum x_i = 1 + 0 = 1$$
$$\theta^* = \frac{\theta}{1 + n\theta} = \frac{0.25}{1 + 2 \times 0.25} = \frac{1}{6}$$
$$E[X_3 | X_1 = 0, X_2 = 0] = \alpha^* \times \theta^* = \frac{1}{6}$$

Solution 10.8

This is the binomial/beta model:

$$f_Q(q) = 4q^3 \quad \text{for } 0 \le q \le 1$$
$$= \frac{\Gamma(5)}{\Gamma(4)\Gamma(1)} q^{4-1}(1-q)^{1-1} \quad \text{(beta with } a = 4, b = 1\text{)}$$

Here we have $m = 4$ trials and $X_1 = 2$, $X_2 = 4$. So the posterior parameters are:

$$a^* = a + \sum x_i = 4 + 6 = 10$$
$$b^* = b + nm - \sum x_i = 1 + 2 \times 4 - 6 = 3$$

The Bayesian credibility estimate of the number of claims in Year 3 is thus:

$$E[X_3 | X_1 = 2, X_2 = 4] = \frac{ma^*}{a^* + b^*} = \frac{4 \times 10}{13} = \frac{40}{13}$$

Solution 10.9

Using the exponential/inverse gamma summary, we have:

$$x_1 = 125,\ x_2 = 500,\ x_3 = 75,\ x_4 = 100\ ,\ \theta = 200\,,\alpha = 2$$

$$\theta^* = \theta + \sum x_i = 200 + 800 = 1{,}000$$

$$\alpha^* = \alpha + n = 2 + 4 = 6$$

$$\Rightarrow \quad E\left[X_4 \mid X_1 = 125\,, X_2 = 500\,,\ X_3 = 75\,,\ X_4 = 100\right]$$

$$= \frac{\theta^*}{\alpha^* - 1} = \frac{1{,}000}{5} = 200$$

Solution 10.10

Using the normal/normal summary, we have:

$$x_1 = 125,\ x_2 = 100,\ x_3 = 325,\ x_4 = 50$$

$$\mu = 100\ ,\ \sigma_1^2 = 100\ ,\ \sigma_2^2 = 225$$

$$\mu^* = \left(\frac{\sum x_i}{\sigma_1^2} + \frac{\mu}{\sigma_2^2}\right) \Bigg/ \left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)$$

$$= \left(\frac{600}{100} + \frac{100}{225}\right) \Bigg/ \left(\frac{4}{100} + \frac{1}{225}\right) = 145$$

$$E\left[X_4 \mid X_1 = 125\,, X_2 = 500\,,\ X_3 = 325\,,\ X_4 = 50\right] = \mu^* = 145$$

Solution 10.11

$$\Pr\left(\text{Type A} \mid S_1 = 300\right) = c\ \underbrace{\Pr\left(\text{Type A}\right)}_{0.5}\ \underbrace{\Pr\left(S_1 = 300 \mid \text{Type A}\right)}_{\substack{\text{3 claims of 100 or 2 claims} \\ \text{equal to 100,200 or 200,100}}}$$

$$= c \times 0.5 \times \left(e^{-0.6} \times \frac{0.6^3}{3!} \times 0.5^3 + 2 \times e^{-0.6} \times \frac{0.6^2}{2!} \times 0.5^2\right)$$

$$= 0.02593c$$

$$\Pr\left(\text{V B} \mid S_1 = 300\right) = c\ \underbrace{\Pr\left(\text{Type B}\right)}_{0.5}\ \underbrace{\Pr\left(S_1 = 300 \mid \text{Type B}\right)}_{\text{1 claim equal to 300}}$$

$$= c \times 0.5 \times \left(e^{-0.4} \times \frac{0.4^1}{1!} \times 0.5\right)$$

$$= 0.06703c$$

$$1 = 0.02593c + 0.06703c \ \Rightarrow\ c = 10.75693$$

$$\Rightarrow \Pr\left(\text{Type A} \mid S_1 = 300\right) = 0.02593c = 0.27894$$

Solution 10.12

The Bayesian credibility estimate for the total annual claim in Year 2 is:

$$\begin{aligned} E[S_2 \mid S_1 = 300] &= E\big[E[S_2 \mid \text{Class}] \mid S_1 = 300\big] \\ &= E[S_2 \mid \text{Class A}] \times \Pr(\text{Class A} \mid S_1 = 300) \\ &\quad + E[S_2 \mid \text{Class B}] \times \Pr(\text{Class B} \mid S_1 = 300) \end{aligned}$$

So we need to determine the expected annual claim for each class and then take a weighted average using the posterior probabilities from Solution 10.11:

$$\begin{aligned} E[S \mid \text{Class A}] &= E[N \mid \text{Class A}]\ E[X \mid \text{Class A}] \\ &= 0.6 \times 150 = 90 \\ E[S \mid \text{Class B}] &= E[N \mid \text{Class B}]\ E[X \mid \text{Class B}] \\ &= 0.4 \times 250 = 100 \end{aligned}$$

Using the results of Solution 10.11, the credibility premium is:

$$\begin{aligned} E[S_2 \mid S_1 = 200] &= \underbrace{E[S_2 \mid \text{Class A}]}_{90} \times \underbrace{\Pr(\text{Class A} \mid S_1 = 300)}_{0.27894} \\ &\quad + \underbrace{E[S_2 \mid \text{Class B}]}_{100} \times \underbrace{\Pr(\text{Class B} \mid S_1 = 300)}_{0.72106} = 97.21 \end{aligned}$$

Solution 10.13

$$\begin{aligned} \Pr(\Theta = 200 \mid X_1 = 350) &= c\Pr(\Theta = 200)\, f_X(350 \mid \Theta = 200) \\ &= c \times 0.8 \times \frac{200}{(200+350)^2} = 0.000529c \\ \Pr(\Theta = 300 \mid X_1 = 350) &= c\Pr(\Theta = 300)\, f_X(350 \mid \Theta = 300) \\ &= c \times 0.2 \times \frac{300}{(300+350)^2} = 0.000142c \end{aligned}$$

So the probability that $\theta = 200$ for this policy is determined as follows:

$$1 = 0.000529c + 0.000142c \Rightarrow c = 1{,}490.45190$$

$$\Rightarrow \Pr(\Theta = 200 \mid X_1 = 350) = 0.000529c = 0.78834$$

Solution 10.14

For the given conditional PDF, the conditional survival function is:

$$f_X(x \mid \Theta=\theta)=\frac{\theta}{(\theta+x)^2} \quad \text{for } x>0$$

$$\Rightarrow s_X(x \mid \Theta=\theta)=\int_x^{\infty}\frac{\theta}{(\theta+t)^2}\,dt = \frac{\theta}{\theta+x}$$

We are asked to calculate $\Pr(X_2>300 \mid X_1=350)$:

$$\begin{aligned}\Pr(X_2>300 \mid X_1=350) &= \sum_{\theta}\Pr(X_2>300 \mid \Theta=\theta)\Pr(\Theta=\theta \mid X_1=350)\\ &= \sum_{\theta} s_X(300 \mid \Theta=\theta)\Pr(\Theta=\theta \mid X_1=350)\\ &= \underbrace{s_X(300 \mid \Theta=200)}_{\frac{200}{200+300}}\underbrace{\Pr(\Theta=200 \mid X_1=350)}_{\text{Solution 10.13: 0.78834}}\\ &\quad+\underbrace{s_X(300 \mid \Theta=300)}_{\frac{300}{300+300}}\underbrace{\Pr(\Theta=300 \mid X_1=350)}_{\text{Solution 10.13: 0.21166}}\\ &= 0.42117\end{aligned}$$

Solution 10.15

$$\begin{aligned}\Pr(\Theta=0.15 \mid X_1=\cdots=X_{10}=0) &= c\Pr(\Theta=0.15)\Pr(X=0 \mid \theta=0.15)^{10}\\ &= c\times 0.2\times e^{-0.15\times 10} = 0.04463c\end{aligned}$$

$$\begin{aligned}\Pr(\Theta=0.30 \mid X_1=\cdots=X_{10}=0) &= c\Pr(\Theta=0.30)\Pr(X=0 \mid \theta=0.30)^{10}\\ &= c\times 0.8\times e^{-0.30\times 10} = 0.03983c\end{aligned}$$

Since the probabilities must sum to one we have:

$$\Pr(\Theta=0.15 \mid X_1=\cdots=X_{10}=0) = 0.04463c = 0.52840$$

Solution 10.16

$$\begin{aligned}\Pr(\Theta=0.15 \mid X_1=\cdots=X_n=0) &= c\Pr(\Theta=0.15)\Pr(X=0 \mid \theta=0.15)^{n}\\ &= c\times 0.2\times e^{-0.15\times n}\end{aligned}$$

$$\begin{aligned}\Pr(\Theta=0.30 \mid X_1=\cdots=X_n=0) &= c\Pr(\Theta=0.30)\Pr(X=0 \mid \theta=0.30)^{n}\\ &= c\times 0.8\times e^{-0.30\times n}\end{aligned}$$

$$\Pr\left(\Theta=0.15 \mid X_1=\cdots=X_n=0\right)=\frac{c\times 0.2\times e^{-0.15\times n}}{c\times 0.2\times e^{-0.15\times n}+c\times 0.8\times e^{-0.30\times n}}$$

$$=\frac{0.2}{0.2+0.8e^{-0.15\times n}}\xrightarrow{n\to\infty}\frac{0.2}{0.2+0.8\times 0}=1$$

Solution 10.17

Using the normal/normal summary, we have:

$$x_1=125,\ x_2=100,\ x_3=325,\ x_4=50$$

$$\mu=100,\ \sigma_1^2=100,\ \sigma_2^2=225$$

$$\mu^*=\left(\frac{\sum x_i}{\sigma_1^2}+\frac{\mu}{\sigma_2^2}\right)\Bigg/\left(\frac{n}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)$$

$$=\left(\frac{600}{100}+\frac{100}{225}\right)\Bigg/\left(\frac{4}{100}+\frac{1}{225}\right)=145$$

$$\sigma^{*2}=1\Bigg/\left(\frac{n}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)=1\Bigg/\left(\frac{4}{100}+\frac{1}{225}\right)=22.50$$

We are asked to determine the posterior probability:

$$\Pr\left(\Theta>150 \mid x_1=125,\ x_2=500,\ x_3=325,\ x_4=500\right)$$

$$=\Pr\left(N\left(\mu^*=145,\ \sigma^{*2}=22.5\right)>150\right)$$

$$=1-\Phi\left(\frac{150-145}{\sqrt{22.5}}\right)=1-\Phi(1.054)=1-0.8541$$

$$=0.1459$$

Solution 10.18

This is the Poisson/gamma model since an exponential with mean 0.25 is a gamma with $\alpha=1,\theta=0.25$:

$$x_1=x_2=0\ \Rightarrow$$

$$\alpha^*=\alpha+\sum x_i=1+0=1$$

$$\theta^*=\frac{\theta}{1+n\theta}=\frac{0.25}{1+2\times 0.25}=\frac{1}{6}$$

So the posterior distribution of the Poisson parameter Λ is a gamma distribution with the above parameters. So in fact it is exponential with mean 1/6. So the posterior probability that $\Lambda>0.25$ is:

$$e^{-0.25/(1/6)}=e^{-1.5}=0.22313$$

Solution 10.19

Step 1. Determine the posterior distribution:

$$\Pr(\Theta=\theta \mid X_1=5) = c\ \Pr(\Theta=\theta)\, f_X(5 \mid \Theta=2) = c\ \Pr(\Theta=\theta)\times\frac{e^{-5/\theta}}{\theta} \ \Rightarrow$$
$$\Pr(\Theta=2 \mid X_1=5) = c\times 0.5\times\frac{e^{-5/2}}{2} = 0.02052c$$
$$\Pr(\Theta=4 \mid X_1=5) = c\times 0.5\times\frac{e^{-5/4}}{4} = 0.03581c$$
$$1 = 0.02052c + 0.03581c \ \Rightarrow\ c = 17.75116 \ \Rightarrow$$
$$\Pr(\Theta=2 \mid X_1=5) = 0.36428$$
$$\Pr(\Theta=4 \mid X_1=5) = 0.63572$$

Step 2. Determine $\Pr(X_2 \geq 10 \mid X_1=5)$:

$$\Pr(X_2 \geq 10 \mid X_1=5) = \sum_\theta \Pr(X_2 > 10 \mid \theta)\ \Pr(\Theta=\theta \mid X_1=5)$$
$$= \sum_\theta e^{-10/\theta}\ \Pr(\Theta=\theta \mid X_1=5)$$
$$= e^{-10/2}\ \Pr(\Theta=2 \mid X_1=5) + e^{-10/4}\ \Pr(\Theta=4 \mid X_1=5)$$
$$= 0.05464$$

Solution 10.20

It helps to recognize that this is the exponential/inverse gamma combination:

exponential PDF: $f_X(x \mid \Lambda=\lambda) = \dfrac{e^{-x/\lambda}}{\lambda}$ for $x>0$

inverse gamma PDF: $f_\Lambda(\lambda) = \dfrac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1}\Gamma(\alpha)} = c\times\dfrac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}}$ for $\lambda>0$

Comparing the inverse gamma PDF with the given formula, we see that the prior is inverse gamma with parameters given as follows:

$$f_\Lambda(\lambda) = e^{-1/\lambda}/\lambda^2 = c\times\frac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}} \ \Rightarrow \theta=\alpha=1$$

We are given $n=1$ observation with $x_1=5$. So according to the exponential/inverse gamma summary, the posterior inverse gamma parameters are:

$$\theta^* = \theta + \sum x_i = 1+5 = 6$$
$$\alpha^* = \alpha + n = 1+1 = 2$$
$$\Rightarrow\ f_\Lambda(\lambda \mid X_1=5) = \frac{6^2\, e^{-6/\lambda}}{\lambda^3} \quad \text{for } \lambda>0$$

The conditional exponential survival function is $\Pr(X > x | \Lambda = \lambda) = e^{-x/\lambda}$, so we have:

$$\Pr(X_2 \geq 10 \mid X_1 = 5) = \int_0^\infty \Pr(X_2 \geq 10 \mid \Lambda = \lambda)\, f_\Lambda(\lambda \mid X_1 = 5)\, d\lambda$$

$$= \int_0^\infty e^{-10/\lambda} \times \frac{6^2 e^{-6/\lambda}}{\lambda^3}\, d\lambda$$

$$= \int_0^\infty \frac{6^2 e^{-16/\lambda}}{\lambda^3}\, d\lambda$$

$$= \frac{6^2}{16^2} \times \underbrace{\int_0^\infty \frac{16^2 e^{-16/\lambda}}{\lambda^3}\, d\lambda}_{\substack{\text{inverse gamma PDF} \\ \theta=16\,,\ \alpha=2}} = \frac{6^2}{16^2} \times 1 = 0.140625$$