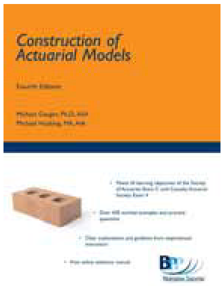

Construction of Actuarial Models

Fourth Edition

by Mike Gauger and Michael Hosking

Published by BPP Professional Education

Solutions to practice questions – Chapter 10

Solution 10.1

From the assumption of a conditional gamma distribution, along with standard gamma moment formulas, we have:

$$\mu(\Theta) = E\left[X|\Theta\right] = 2\Theta \quad , \quad v(\Theta) = \operatorname{var}\left(X|\Theta\right) = 2\Theta^2$$

From the assumption that Θ is uniformly distributed on $\left[50,100\right]$, along with standard uniform moment formulas, we have:

$$\mu = E[X] = E\left[\mu(\Theta)\right] = E[2\Theta] = 2\times 75 = 150$$

$$v = E\left[\operatorname{var}(X|\Theta)\right] = E\left[2\Theta^2\right] = 2\left(75^2 + \frac{50^2}{12}\right) = 11{,}666.67$$

$$a = \operatorname{var}\left(\mu(\Theta)\right) = \operatorname{var}(2\Theta) = 4\times\frac{50^2}{12} = 833.33$$

Solution 10.2

From the assumption of a conditional gamma distribution, along with standard gamma moment formulas, we have:

$$\mu(\Theta) = E\left[X|\Theta\right] = 2\Theta \quad , \quad v(\Theta) = \operatorname{var}\left(X|\Theta\right) = 2\Theta^2$$

From the assumption that Θ is either 25 or 100 with respective probabilities 0.6 and 0.4, we have:

$$E[\Theta] = 55 \quad , \quad E\left[\Theta^2\right] = 4{,}375$$

The remaining structural parameters are:

$$\mu = E[X] = E\left[\mu(\Theta)\right] = E[2\Theta] = 2\times 55 = 110$$

$$v = E\left[\operatorname{var}(X|\Theta)\right] = E\left[2\Theta^2\right] = 2\times 4{,}375 = 8{,}750$$

$$a = \operatorname{var}\left(\mu(\Theta)\right) = \operatorname{var}(2\Theta) = 4\times\left(4{,}375 - 55^2\right) = 5{,}400$$

Solution 10.3

From the given data we have $\bar{x}_3 = (125+240+90)/3 = 151.67$

From the results in Solution 10.1, we have the following credibility prediction:

$$Z = \frac{n}{n+(v/a)} = \frac{3}{3+14} = \frac{3}{17}$$
$$Z\bar{x}_3 + (1-Z)E[X] = \frac{3}{17} \times 151.67 + \frac{14}{17} \times 150 = 150.29$$

Solution 10.4

From the given data we have $\bar{x}_3 = (125+240+90)/3 = 151.67$

From the results in Solution 10.2, we have the following credibility prediction:

$$Z = \frac{n}{n+(v/a)} = \frac{3}{3+1.62037} = 0.64930$$
$$Z\bar{x}_3 + (1-Z)E[X] = 0.64930 \times 151.67 + 0.35070 \times 110 = 137.05$$

Solution 10.5

$$\frac{2}{3} = Z_{100} = \frac{100}{100+(v/a)} \Rightarrow v/a = 50$$
$$\Rightarrow Z_{200} = \frac{200}{200+(v/a)} = \frac{200}{250} = 0.80$$

Solution 10.6

In Solution 10.5 we saw that $v/a = 50$. So to attain 90% credibility we must have:

$$0.90 = Z = \frac{n}{n+(v/a)} = \frac{n}{n+50} \Rightarrow n = 450$$

Solution 10.7

Since they have the same risk parameter the covariance between N_1 and N_2 is the same as Bühlmann's a, the variance of the hypothetical means (see Section 10.2). From the conditional negative binomial assumptions and standard moment formulas, we have:

$$\mu(\Theta) = E[N|\Theta] = 2\Theta$$

Since Θ is assumed to be uniform on $\left[5,10\right]$, we have:

$$\operatorname{var}(\Theta) = \frac{5^2}{12} = \frac{25}{12}$$

Finally, we have:

$$\operatorname{cov}(N_1, N_2) = a = \operatorname{var}(\mu(\Theta)) = \operatorname{var}(2\Theta) = 4 \times \frac{25}{12} = \frac{25}{3}$$

Solution 10.8

According to a result in Section 10.2, $\operatorname{var}(\bar{N}_n) = \frac{v}{n} + a$. Here we have $n = 2$ observations. In Solution 10.7 we saw that $a = 25/3$. So we must compute v:

$$v(\Theta) = \operatorname{var}(N \mid \Theta) = 2\Theta(1+\Theta)$$
$$\Rightarrow v = E\left[v(\Theta)\right] = E\left[2\Theta(1+\Theta)\right] = 2\times\left(E[\Theta] + E\left[\Theta^2\right]\right)$$
$$= 2 \times \left(7.5 + \left(7.5^2 + \frac{5^2}{12}\right)\right) = 131.67$$
$$\Rightarrow \operatorname{var}(\bar{N}_2) = \frac{v}{2} + a = 65.83 + \frac{25}{3} = 74.17$$

Solution 10.9

According to a result in Section 10.2, the ESE is equal to $v + (1-Z)a$. From Solutions 10.1 and 10.3 we have:

$$v = 11{,}666.67 \ , \ a = 833.33 \ , \ Z = \frac{3}{17}$$
$$\Rightarrow ESE = v + (1-Z)a = 12{,}352.94$$

Solution 10.10

$$\mu(\Lambda) = E\left[N \mid \Lambda\right] = \Lambda \quad \text{(Poisson formula)}$$
$$v(\Lambda) = \operatorname{var}(N \mid \Lambda) = \Lambda \quad \text{(Poisson formula)}$$
$$\mu = E[N] = E\left[\mu(\Lambda)\right] = E[\Lambda] = \alpha\theta = 2\times 0.15 = 0.30 \quad \text{(gamma formula)}$$
$$v = E\left[v(\Lambda)\right] = E[\Lambda] = 0.30 \quad \text{(same as above)}$$
$$a = \operatorname{var}(\mu(\Lambda)) = \operatorname{var}(\Lambda) = \alpha\theta^2 = 2 \times 0.15^2 = 0.045 \quad \text{(gamma formula)}$$
$$Z = \frac{n}{n + (v/a)} = \frac{4}{4 + (0.30/0.045)} = 0.375$$
$$\text{prediction} = Z\bar{n}_4 + (1-Z)E[N] = 0.375\times\frac{0}{4} + 0.625\times 0.30 = 0.1875$$

Note: This is the Poisson/gamma model where the Bayesian credibility estimate is linear. So you could also solve this problem from memorized Bayesian results for this model. See Chapter 9.

Solution 10.11

From (i) and (ii) we have:

$$\nu(\Lambda) = \text{var}(N \mid \Lambda) = \Lambda \quad \text{(Poisson formula)}$$
$$\nu = E\left[\nu(\Lambda)\right] = E[\Lambda] = 0.30$$

There is not enough information to compute $a = \text{var}\left(E\left[N|\Lambda\right]\right) = \text{var}(\Lambda)$ directly. However, we can compute $\text{var}(N)$ from (iii), and we know that $\text{var}(N) = \nu + a$:

$$\text{(iii)} \Rightarrow \text{var}(N) = r\beta(1+\beta) = 2\times 0.15\times 1.15 = 0.345$$
$$0.345 = \text{var}(N) = \nu + a = 0.30 + a \Rightarrow a = 0.045$$
$$\Rightarrow k = \nu / a = 0.30/0.045 = 6.667 \Rightarrow Z = \frac{3}{3+6.667} = 0.3103$$

Solution 10.12

Note first that the distribution of Θ is beta with parameters $a=3, b=1$. So from beta moment formulas we have:

$$E[\Theta] = \frac{a}{a+b} = \frac{3}{4} \quad , \; E\left[\Theta^2\right] = \frac{a}{a+b}\times\frac{a+1}{a+b+1} = \frac{3}{4}\times\frac{4}{5} = \frac{3}{5}$$

The structural parameters are:

$$\mu(\Theta) = E\left[N \mid \Theta\right] = 3\Theta \quad \text{(binomial formula)}$$
$$\nu(\Theta) = \text{var}(N \mid \Theta) = 3\Theta(1-\Theta) \quad \text{(binomial formula)}$$
$$\mu = E[N] = E\left[\mu(\Theta)\right] = E[3\Theta] = \frac{9}{4} \quad \text{(beta formula)}$$
$$\nu = E\left[\nu(\Theta)\right] = E\left[3\Theta(1-\Theta)\right] = 3\times\left(\frac{3}{4}-\frac{3}{5}\right) = 0.45 \quad \text{(beta formulas)}$$
$$a = \text{var}\left(\mu(\Theta)\right) = \text{var}(3\Theta) = 9\,\text{var}(\Theta) = 9\left(\frac{3}{5}-\left(\frac{3}{4}\right)^2\right) = 0.3375 \quad \text{(beta formula)}$$
$$Z = \frac{n}{n+(\nu/a)} = \frac{2}{2+(0.45/0.33750)} = 0.6$$
$$\text{prediction} = Z\bar{n}_2 + (1-Z)E[N] = 0.60\times\frac{6}{2} + 0.40\times\frac{9}{4} = 2.7$$

Note: This is the binomial/beta model where the Bayesian credibility estimate is linear. So you could also solve this problem from memorized Bayesian results for this model. See Chapter 9.

Solution 10.13

$\mu(\Lambda) = E\left[X \mid \Lambda\right] = \Lambda$ (exponential formula)

$\nu(\Lambda) = \text{var}\left(X \mid \Lambda\right) = \Lambda^2$ (exponential formula)

$\mu = E[X] = E\left[\mu(\Lambda)\right] = E[\Lambda] = \dfrac{\theta}{\alpha-1} = \dfrac{400}{2} = 200$ (Pareto formula)

$\nu = E\left[\nu(\Lambda)\right] = E\left[\Lambda^2\right] = \dfrac{2\theta^2}{(\alpha-1)(\alpha-2)} = \dfrac{2\times 400^2}{2} = 160,000$ (Pareto formula)

$a = \text{var}\left(\mu(\Lambda)\right) = \text{var}(\Lambda) = 160,000 - 200^2 =$ 120,000 (Pareto formula)

$Z = \dfrac{n}{n+(\nu/a)} = \dfrac{4}{4+(4/3)} = 0.75$

prediction $= Z\bar{x}_4 + (1-Z)E[X] = 0.75\times\dfrac{1,210}{4} + 0.25\times 200 = 276.88$

Solution 10.14

From (ii) we can see that the distribution of the risk parameter Λ is a two-point mixture of two gamma distributions:

30% gamma with parameters $\alpha = 2, \theta = 1/7$
70% gamma with parameters $\alpha = 2, \theta = 1/3$

For a gamma distribution, we have:

first moment = $\alpha\theta$, second moment = $\alpha(\alpha+1)\theta^2$

So the first two moments for our 2-point mixture are:

$$E[\Lambda] = 0.3\times\left(2\times\frac{1}{7}\right) + 0.7\times\left(2\times\frac{1}{3}\right) = 0.55238$$

$$E\left[\Lambda^2\right] = 0.3\times\left(2\times 3\times\left(\frac{1}{7}\right)^2\right) + 0.7\times\left(2\times 3\times\left(\frac{1}{3}\right)^2\right) = 0.50340$$

The structural parameters are:

$\mu(\Lambda) = E\left[X \mid \Lambda\right] = \Lambda$ (Poisson formula)

$\nu(\Lambda) = \text{var}\left(X \mid \Lambda\right) = \Lambda$ (Poisson formula)

$\mu = E[X] = E\left[\mu(\Lambda)\right] = E[\Lambda] = 0.55238$ (2-point gamma mix)

$\nu = E\left[\nu(\Lambda)\right] = E[\Lambda] = 0.55238$ (2-point gamma mix)

$a = \text{var}\left(\mu(\Lambda)\right) = \text{var}(\Lambda) = 0.50340 - 0.55238^2 = 0.19828$ (2-point gamma mix)

$Z = \dfrac{n}{n+(\nu/a)} = \dfrac{2}{2+2.78591} = 0.41789$

prediction $= Z\bar{x}_2 + (1-Z)E[X] = 0.41789\times\dfrac{0}{2} + 0.58211\times 0.55238 = 0.32154$

Solution 10.15

In Section 10.1 we indicated that by substituting $m=1$ into the formulas for the averages Y_i we could obtain formulas for the hidden underlying $X's$, thus turning a Bühlmann-Straub problem into an ordinary Bühlmann problem. Substituting $m=1$ results in our old friend from Chapter 9, the normal/normal model, where the Bayesian prediction is a linear function of the data. Recall that this means the Bayesian and Bühlmann predictions are identical. So let's make this one quick and use a Chapter 9 formula for the Bayesian credibility prediction.

For the normal/normal model, we are given:

$$\mu = 200\ ,\ \sigma_1^2 = 275\ ,\ \sigma_2^2 = 625$$
$$n = 20+10 = 30$$
$$\bar{x}_{30} = \frac{20\times 225 + 10\times 210}{30} = 220$$
$$\mu^* = \left(\frac{\sum x_i}{\sigma_1^2} + \frac{\mu}{\sigma_2^2}\right)\bigg/\left(\frac{n}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) = \left(\frac{30\times 220}{275} + \frac{200}{625}\right)\bigg/\left(\frac{30}{275} + \frac{1}{625}\right) = 219.71$$

linerar Bayesian $\Rightarrow$ Bühlmann $\Rightarrow$ Bayesian $=\mu^*$ (posterior mean) $= 219.71$

Solution 10.16

From the first two columns of the joint probability table we can determine the conditional distributions of X for $\theta=0$ and $\theta=1$:

X	$\Pr(X=x \mid \theta=0)$	$\Pr(X=x \mid \theta=1)$
0	3/7	3/6
1	2/7	2/6
2	2/7	1/6

From this table we can construct a table of values of the conditional means and variances:

Θ	$\Pr(\Theta=\theta)$	$E[X\|\Theta]$	$\operatorname{var}(X \mid \Theta)$
0	0.7	6/7	34/49
1	0.3	4/6	20/36

The marginal probabilities for the risk parameter in the table above are obtained by summing down the columns in the joint probability table.

Solution 10.17

From the results in Solution 10.16, we have:

$$\mu = E[X] = E\left[E\left[X|\Theta\right]\right] = 0.7 \times \frac{6}{7} + 0.3 \times \frac{4}{6} = 0.8$$

$$\nu = E\left[\text{var}\left(X|\Theta\right)\right] = 0.7 \times \frac{34}{49} + 0.3 \times \frac{20}{36} = 0.65238$$

$$a = \text{var}\left(E\left[X \mid \Theta\right]\right) = \left(0.7 \times \left(\frac{6}{7}\right)^2 + 0.3 \times \left(\frac{4}{6}\right)^2\right) - 0.80^2 = 0.00762$$

$$Z = \frac{n}{n + (\nu / a)} = \frac{10}{10 + 85.625} = 0.10458$$

Note: We could have obtained the marginal distribution of X by summing across the rows of the joint probability table. This would provide an alternative way to determine the overall mean $E[X]$ and a way to check the computations of ν and a since $\text{var}(X) = \nu + a$.

Solution 10.18

From the results of Solutions 10.16 and 10.17, we have:

$$\text{prediction} = Z\bar{x}_{10} + (1 - Z)E[X] = 0.10458 \times \frac{4}{10} + 0.89542 \times 0.8 = 0.75817$$

Solution 10.19

From the conditional PDF for the annual claim amount, we have:

$$f_X(x \mid \Theta = \theta) = \frac{2(\theta - x)}{\theta^2} \quad \text{for } 0 \le x \le \theta$$

$$\Rightarrow \quad E[X \mid \Theta] = \int_0^\Theta x \times \frac{2(\Theta - x)}{\Theta^2}\,dx = \frac{\Theta}{3}$$

$$E\left[X^2 \mid \Theta\right] = \int_0^\Theta x^2 \times \frac{2(\Theta - x)}{\Theta^2}\,dx = \frac{\Theta^2}{6}$$

$$\operatorname{var}(X \mid \Theta) = \frac{\Theta^2}{18}$$

From the distribution of Θ and the results above, we have:

$$\pi(\theta) = \frac{2\theta}{7500} \quad \text{for } 50 \le \theta \le 100$$

$$\Rightarrow \quad \mu = E[X] = E\big[E[X \mid \Theta]\big] = E\left[\frac{\Theta}{3}\right] = \frac{1}{3} \times \int_{50}^{100} \theta \times \frac{2\theta}{7500}\,d\theta = 25.92593 \quad (\Rightarrow E[\Theta] = 77.77778)$$

$$v = E\big[\operatorname{var}(X \mid \Theta)\big] = E\left[\frac{\Theta^2}{18}\right] = \frac{1}{18} \times \int_{50}^{100} \theta^2 \times \frac{2\theta}{7500}\,d\theta = 347.22222 \quad \left(\Rightarrow E\left[\Theta^2\right] = 6{,}250\right)$$

$$a = \operatorname{var}\big(E[X \mid \Theta]\big) = \operatorname{var}\left(\frac{\Theta}{3}\right) = \frac{\operatorname{var}(\Theta)}{9} = \frac{6{,}250 - (77.77778)^2}{9} = 22.29081$$

Solution 10.20

From the results of Solution 10.19, we have:

$$Z = \frac{1}{1 + (v/a)} = \frac{1}{1 + (15.57692)} = 0.06032$$

$$\text{prediction } = Z\bar{x}_1 + (1 - Z)E[X] = 0.06032 \times 50 + 0.93968 \times 25.92593 = 27.38$$