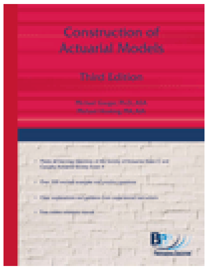

Construction of Actuarial Models

Third Edition

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Solutions to practice questions – Chapter 1

Solution 1.1

$$M_X(t) = (1-10t)^{-2} \quad \Rightarrow \quad M'_X(t) = 20(1-10t)^{-3} \;,\; M''_X(t) = 600(1-10t)^{-4}$$

$$E[X] = M'_X(0) = 20 \;,\; E\left[X^2\right] = M''_X(0) = 600 \;, \operatorname{var}(X) = 600 - 20^2 = 200$$

Solution 1.2

$$M_N(t) = E\left[e^{tN}\right] = \sum_{k=0}^{4} e^{tk} \Pr(N=k) = \sum_{k=0}^{4} e^{tk} \binom{4}{k} 0.7^k \, 0.3^{4-k}$$

$$= \sum_{k=0}^{4} \binom{4}{k} \left(0.7e^t\right)^k 0.3^{4-k} = \left(0.7e^t + 0.3\right)^4$$

binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k}$

$$P_N(t) = M_N(\ln(t)) = \left(0.7\, e^{\ln(t)} + 0.3\right)^4 = (0.7\,t + 0.3)^4$$

Solution 1.3

$$P_{N_1+N_2}(t) = E\left[t^{N_1+N_2}\right] = E\left[t^{N_1}\right] E\left[t^{N_2}\right] = P_{N_1}(t) P_{N_2}(t) = (0.7t + 0.3)^8$$

Solution 1.4

The possible values of $X_1 + X_2 + X_3 + X_4$ are: $4, 5, 6, 7, 8$. Here are the first few probabilities:

$$f_X^{*4}(4) = f_X^{*3}(3)\, f_X(1) = 0.2401$$

$$f_X^{*4}(5) = f_X^{*3}(3)\, f_X(2) \;+\; f_X^{*3}(4)\, f_X(1) = 0.4116$$

We leave it to the reader to check the following:

$$f_X^{*4}(6) = 0.2646 \;,\; f_X^{*4}(7) = 0.0756 \;,\; f_X^{*4}(8) = 0.0081$$

Solution 1.5

We have $S = X_1 + \cdots + X_{100}$ where the various X_i are independent and identically distributed like X:

$$f_X(x) = \begin{cases} 0.80 & x = 0 \text{ (discrete part)} \\ 0.20/1900 & 100 \le x \le 2000 \text{ (continuous part)} \end{cases}$$

$$E\left[X^k\right] = 0.8 \times 0^k + 0.2 \int_{100}^{2,000} x^k \frac{1}{1,900} dx = 0.2 \left(\frac{2,000^{k+1} - 100^{k+1}}{1,900(k+1)} \right) \Rightarrow$$

$$E[X] = 210 \ , \ E\left[X^2\right] = 280,666.66 \ , \ \text{var}(X) = 236,566.67$$

Now move up to the aggregate level:

$$E[S] = 100\,E[X] = 21,000$$

$$\text{var}(S) = 100\,\text{var}(X) = 23,656,666.67$$

$$95\text{-}th \text{ percentile} = E[S] + 1.645\sqrt{\text{var}(S)} = 29,001$$

Solution 1.6

Let Y be the height and let I be an indicator for the sex of an individual:

$$I = \begin{cases} 0 & \text{male} \\ 1 & \text{female} \end{cases} , \text{ where } \Pr(I=0) = 0.45 \ , \Pr(I=1) = 0.55$$

The given data can be interpreted as being conditional means and variances for the two groups:

$I = i$	$\Pr(I = i)$	$E[Y \mid I = i]$	$\text{var}(Y \mid I = i)$
0	0.45	70	$4^2 = 16$
1	0.55	67	$3^2 = 9$

Now apply the double expectation theorem:

$$E[Y] = E\left[E[Y|I]\right] = 0.45 \times 70 + 0.55 \times 67 = 68.35$$

$$E\left[\text{var}(Y|I)\right] = 0.45 \times 16 + 0.55 \times 9 = 12.15$$

$$\text{var}\left(E[Y|I]\right) = E\left[\left(E[Y|I]\right)^2\right] - \left(E\left[E[Y \mid I]\right]\right)^2$$

$$= 0.45 \times 70^2 + 0.55 \times 67^2 - 68.35^2 = 2.22750$$

$$\text{var}(Y) = 12.15 + 2.22750 = 14.3775$$

So the standard deviation of Y is 3.79.

Solution 1.7

IRM: $X_1 = 140 + 30 = 170$, $X_2 = 57 + 90 = 147$, $X_3 = 0$, $X_4 = 100$
$S = X_1 + X_2 + X_3 + X_4 = 417$

CRM: $N = 5$, $Y_1 = 57$, $Y_2 = 100$, $Y_3 = 90$, $Y_4 = 140$, $Y_5 = 30$
$S = Y_1 + \cdots + Y_N = Y_1 + \cdots + Y_5 = 417$

Solution 1.8

$$E[N] = \text{var}(N) = \lambda = 5 \quad , \quad E[Y] = \theta = 1/0.002 = 500 \, , \ \text{var}(Y) = \theta^2 = 250{,}000$$

$$E[S] = E[N]\,E[Y] = 5 \times 500 = 2{,}500$$
$$\text{var}(S) = E[N]\,\text{var}(Y) + \left(E[Y]\right)^2 \text{var}(N) = 2{,}500{,}000$$

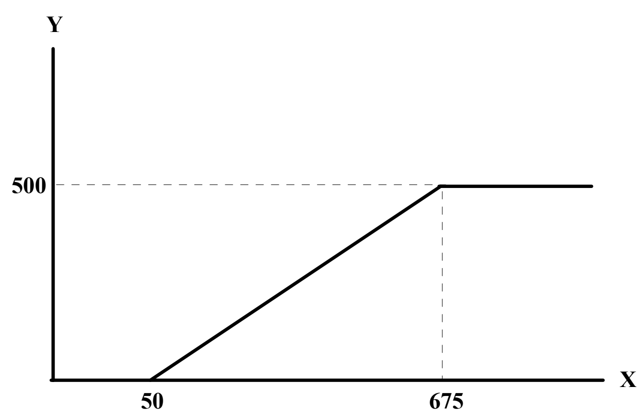
$$M_N(t) = e^{\lambda\left(e^t - 1\right)} = e^{5\left(e^t - 1\right)} \, , \ M_Y(t) = (1 - \theta t)^{-1} = (1 - 500t)^{-1}$$
$$\Rightarrow \ M_S(t) = M_N\left(\ln\left(M_Y(t)\right)\right) = e^{5\left(M_Y(t) - 1\right)} = e^{5\left((1-500t)^{-1} - 1\right)} \quad \text{for } t < 1/500$$

$\Pr(S = 0) = \Pr(N = 0) = e^{-5}$ since Y is continuous

Solution 1.9

$$Y = \begin{cases} 0 & \text{if } X \le 50 \\ 0.8(X - 50) & \text{if } 50 < X \le 675 \\ 500 & \text{if } 675 < X \end{cases}$$

since $500 = 0.8(X - 50)$ when $X = 675$.



Solution 1.10

From the formula in Solution 1.9, we have:

$$F_Y(y) = \Pr(Y \le y) = 0 \quad \text{when } y < 0$$

For y in the range $[0\ ,\ 500)$, we have:

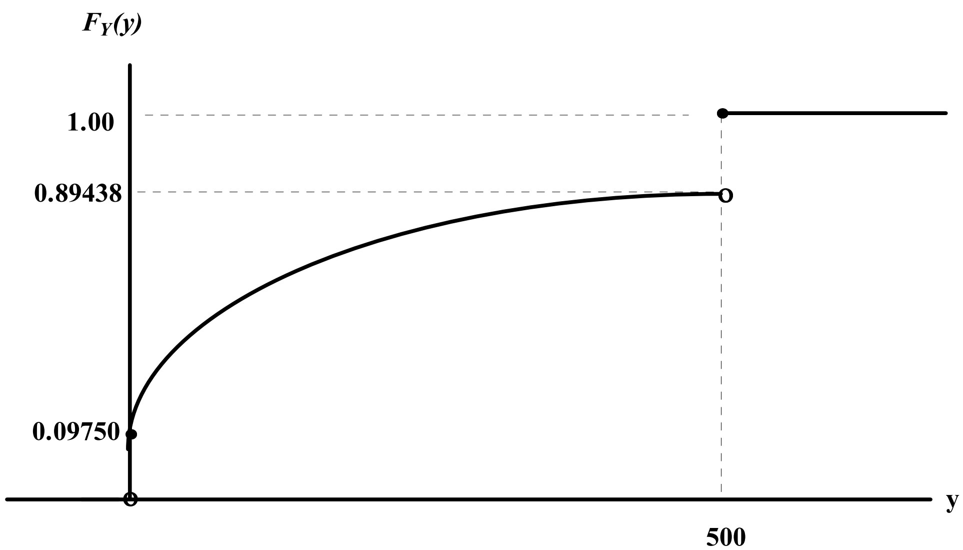
$$F_Y(y) = \Pr(Y \le y) = \Pr\big(0.8(X-50) \le y\big) = \Pr(X \le 1.25y + 50)$$
$$= \int_0^{1.25y+50} \frac{2(1{,}000-x)}{1{,}000^2}\,dx = 1 - \left(\frac{760-y}{800}\right)^2$$

For $y \ge 500$ we have:

$$F_Y(y) = 1$$

since the largest possible value of Y is 500.

Here is the graph:



Solution 1.11

The discrete part is concentrated at the jump discontinuities:

$$f_Y(y) = \begin{cases} 1-\left(\dfrac{760}{800}\right)^2 = 0.09750 & \text{when } y=0 \quad \text{(discrete part)} \\ \left(\dfrac{260}{800}\right)^2 = 0.10563 & \text{when } y=500 \quad \text{(discrete part)} \\ F'(y) = \dfrac{2(760-y)}{800^2} & \text{when } 0 < y < 500 \quad \text{(continuous part)} \end{cases}$$

Solution 1.12

$$E[Y] = 0\times(0.09750) + 500\times(0.10563) + \int_0^{500} y\times\frac{2(760-y)}{800^2}\,dy$$

$$= 52.81250 + \left(\frac{\left(760y^2 - 2y^3/3\right)}{800^2}\right)\Bigg|_0^{500} = 52.81250 + 166.67 = 219.48$$

Solution 1.13

$$E[Y] = \int_0^{50} 0\times f_X(x)\,dx + \int_{50}^{675} 0.8(x-50)\,f_X(x)\,dx + \int_{675}^{1,000} 500\,f_X(x)\,dx$$

$$= \int_{50}^{675} 0.8(x-50)\times\frac{2(1,000-x)}{1,000^2}\,dx + \int_{675}^{1,000} 500\times\frac{2(1,000-x)}{1,000^2}\,dx$$

$$= \int_0^{500} u\times\frac{2(760-u)}{800^2}\,du + 500\times\left(\frac{325}{1,000}\right)^2 \text{(substitute } u=0.8(x-50))$$

$$= \frac{760\times500^2 - 2\times500^3/3}{800^2} + 500\times\left(\frac{325}{1,000}\right)^2 = 219.48$$

Solution 1.14

The conditional densities are uniform:

$$f_X(x \mid M=m) = \frac{1}{m} \quad \text{for} \quad 0 \le x \le m$$

The marginal density function for M is:

$$f_M(1,000) = 0.75 \quad , \quad f_M(2,000) = 0.25$$

So the marginal density function for a randomly selected loss is:

$$f_X(x) = \sum_m f_{X,M}(x,m) = \sum_m f_M(m)\;f_X(x \mid m)$$
$$= 0.75\,f_X(x \mid m=1,000) + 0.25\,f_X(x \mid m=2,000)$$

$$= 0.75\times\begin{cases} 0.001 & \text{when } 0 \le x \le 1,000 \\ 0 & \text{otherwise} \end{cases} + 0.25\times\begin{cases} 0.0005 & \text{when } 0 \le x \le 2,000 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0.000875 & \text{when } 0 \le x \le 1,000 \\ 0.000125 & \text{when } 1,000 < x \le 2000 \end{cases}$$

Solution 1.15

We are given that $X | M = m$ is uniform on the interval $[0, m]$. Also, the distribution of M is:

$$\Pr(M = 1{,}000) = 0.75 \quad \text{and} \quad \Pr(M = 2{,}000) = 0.25$$

As a result, we have:

$$E\left[X^k | M = m\right] = \int_0^m x^k \cdot \frac{1}{m} dx = \frac{m^k}{k+1}$$

So the conditional mean and conditional variance are:

$$E[X | M] = \frac{M}{2} \quad , \quad \text{var}(X | M) = \frac{M^2}{3} - \left(\frac{M}{2}\right)^2 = \frac{M^2}{12}$$

Now apply the double expectation theorem:

(i) $$E[X] = E\big[E[X | M]\big] = E\left[\frac{M}{2}\right] = \frac{1{,}000 \times 0.75 + 2{,}000 \times 0.25}{2} = 625$$

(ii)
$$\begin{aligned}
\text{var}(X) &= E\big[\text{var}(X | M)\big] + \text{var}\big(E[X | M]\big) \\
&= E\left[\frac{M^2}{12}\right] + \text{var}\left(\frac{M}{2}\right) = \frac{E\left[M^2\right]}{12} + \frac{E\left[M^2\right] - \left(E[M]\right)^2}{4} \\
&= \frac{4E\left[M^2\right] - 3\left(E[M]\right)^2}{12} \\
&= \frac{4\left(1{,}000^2 \times 0.75 + 2{,}000^2 \times 0.25\right) - 3 \times \left(1{,}000 \times 0.75 + 2{,}000 \times 0.25\right)^2}{12} \\
&= 192{,}708.33
\end{aligned}$$

Solution 1.16

We have:

$$\Pr(N \geq 2) = 1 - \underbrace{\Pr(N = 0)}_{\substack{0.56356 \\ \text{Ex. 9.15}}} - \underbrace{\Pr(N = 1)}_{\substack{0.30840 \\ \text{below}}} = 0.12804$$

$$\begin{aligned}
f_N(1) &= \int f_{N,\Theta}(1, \theta) \, d\theta = \int f_\Theta(\theta) \, f_N(1 | \Theta = \theta) \, d\theta \\
&= \int_{0.2}^{1.0} \frac{1}{0.8} e^{-\theta} \theta \, d\theta = \left(-1.25\left(e^{-\theta}(1 + \theta)\right)\right)\Big|_{0.2}^{1.0} = 0.30840
\end{aligned}$$

Solution 1.17

Using the method of transformations, we have:

$$y = x^{-1} \;\Rightarrow\; x = y^{-1} \;\Rightarrow\; \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_X(x) = \frac{e^{-x/50}}{50} \quad \text{for } x > 0$$

$$f_Y(y) = f_X\left(y^{-1}\right)\left|\frac{dx}{dy}\right| = \frac{e^{-1/50y}}{50y^2} \quad \text{for } y > 0$$

Solution 1.18

For the spliced distribution, we have:

$$f(x) = \begin{cases} 0.8\, f_1(x) & \text{when } 0 < x \le 1{,}000 \\ 0.2\, f_2(x) & \text{when } x > 1{,}000 \end{cases}$$

$$= \begin{cases} 0.0008 & \text{when } 0 < x \le 1{,}000 \\ \dfrac{0.4 \times 1{,}000^2}{x^3} & \text{when } x > 1{,}000 \end{cases}$$

$$E[X] = \int_0^{1{,}000} 0.0008x\, dx + \int_{1{,}000}^{\infty} \frac{0.4 \times 1{,}000^2\, x}{x^3}\, dx$$

$$= 400 + \left(-\frac{400{,}000}{x}\right)\Bigg|_{1{,}000}^{\infty} = 400 + 400 = 800$$

Solution 1.19

Since $Y = X \mid X > 100$, we have:

$$f_Y(y) = \frac{f_X(y)}{\Pr(X > 100)} \quad \text{for } 100 < y < \infty$$

$$= \frac{e^{-y/500}/500}{e^{-100/500}} \quad \text{for } 100 < y < \infty$$

$$= \frac{\exp\left(-\left(\dfrac{y-100}{500}\right)\right)}{500} \quad \text{for } 100 < y < \infty$$

The expected value could be computed in several ways:

- $$E[Y] = \int_{100}^{\infty} y \exp\left(-\left(\frac{y-100}{500}\right)\right) dy / 500$$

- $$Y = X \mid X > 100 = \underbrace{(Y-100) \mid X > 100}_{\substack{\text{exponential with mean} \\ \text{500 due to memoryless} \\ \text{property of exponentials}}} + 100$$

  $$\Rightarrow \ E[Y] = 500 + 100 = 600$$

Solution 1.20

The pdf is:

$$f_Y(y) = \begin{cases} f_X(y) & \text{when } y < 1{,}000 \\ s_X(1{,}000) & \text{when } y = 1{,}000 \text{ (discrete part)} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-y/500}}{500} & \text{when } y < 1{,}000 \\ e^{-1{,}000/500} & \text{when } y = 1{,}000 \text{ (discrete part)} \end{cases}$$

The expected value could be obtained by evaluating the integral:

$$E[Y] = \int_0^{1{,}000} y \frac{e^{-y/500}}{500} dy + 1{,}000 e^{-2}$$

In life contingencies (Exam M) this expected value is the same as:

$$\mathring{e}_{0:\overline{1{,}000|}} = \int_0^{1{,}000} s_X(x) dx = \int_0^{1{,}000} e^{-x/500} dx = \left(-500e^{-x/500}\right)\Big|_0^{1{,}000}$$

$$= 500\left(1-e^{-2}\right) = 432.33$$

